Bayesian Constraints of Quark Gluon Plasma Properties

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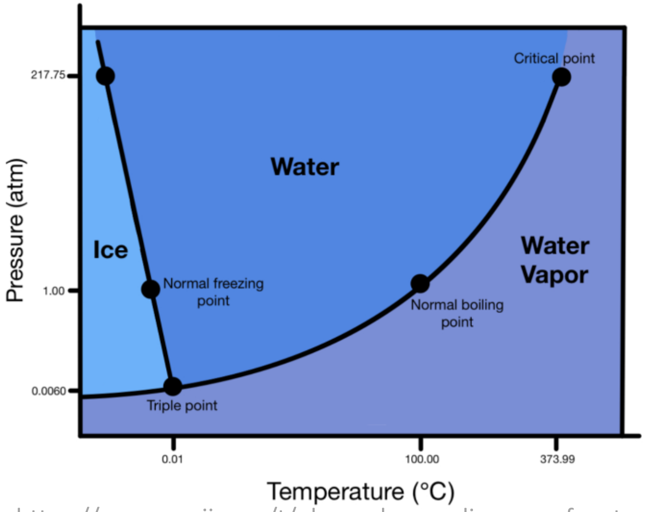
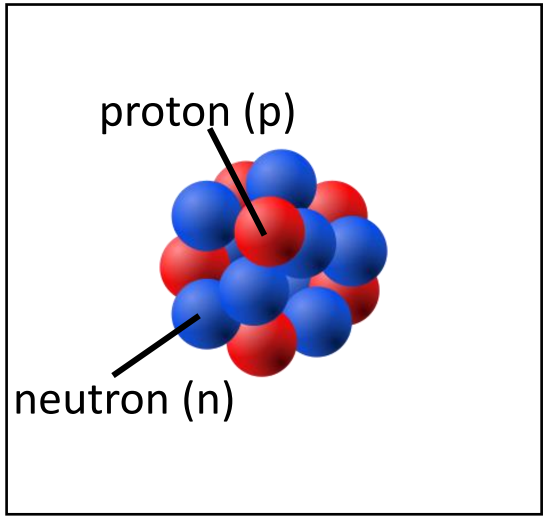
High energy nuclear collisions & nuclear equation of state

$T \uparrow$



Ref.: https://www.expii.com/t/phase-change-diagram-of-water-overview-importance-8031

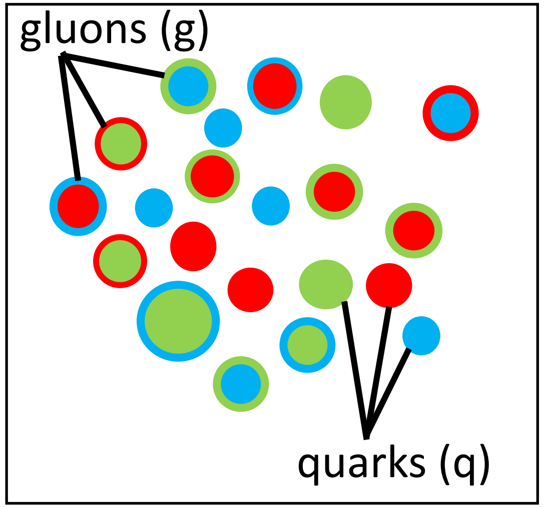
Ref.: https://en.wikipedia.org/wiki/Atomic_nucleus



$T \sim 10^2\ Kelvin\ (10^{-2}\ eV)$

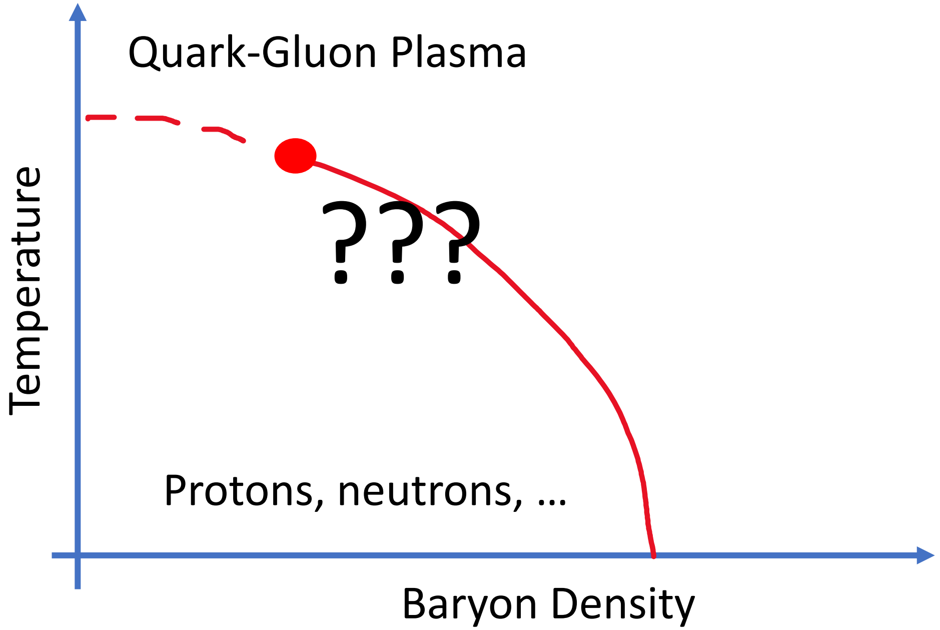
nucleus

$T \uparrow$

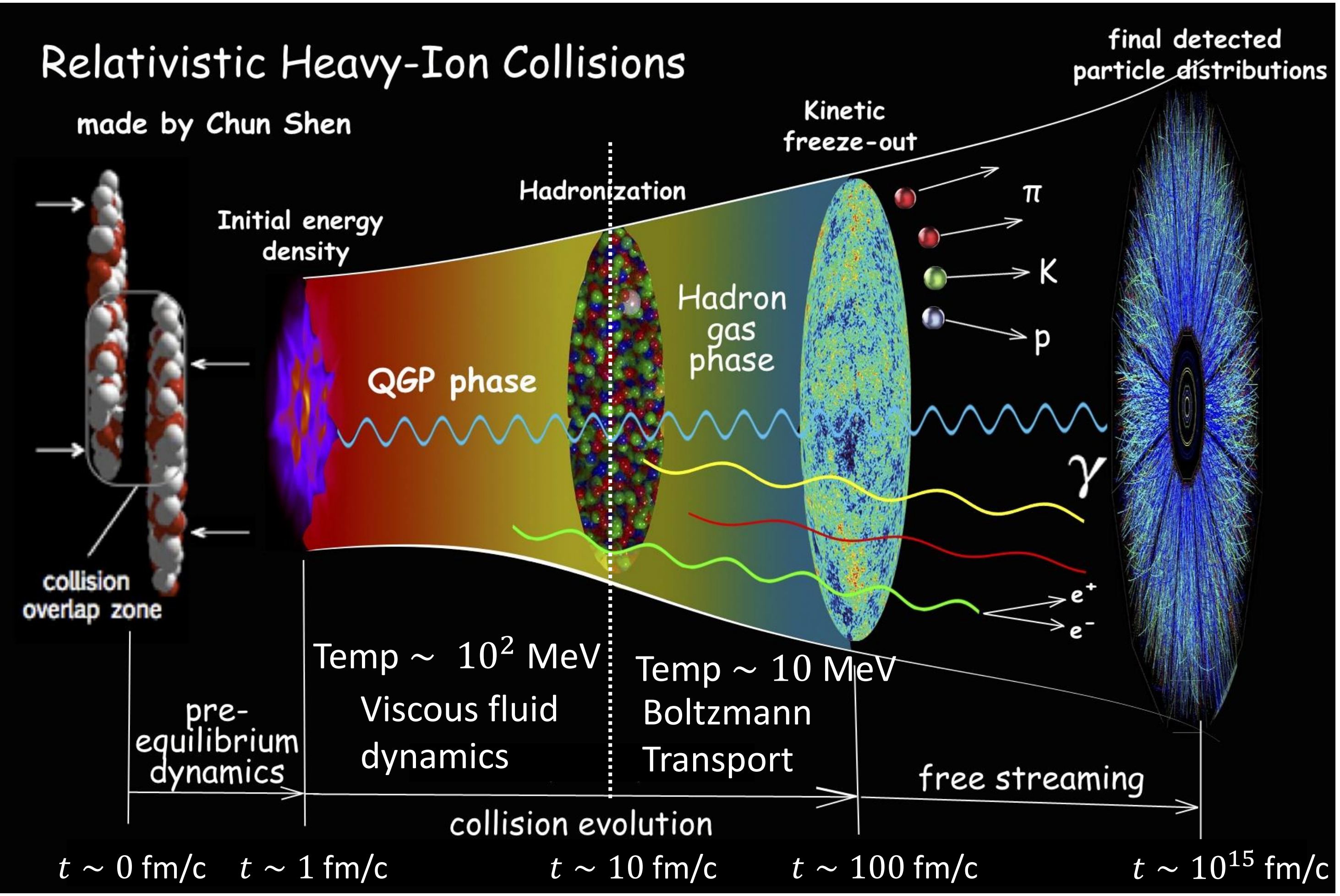


$T \sim 10^{12}\ Kelvin\ (\sim 10^8\ eV)$

Quark Gluons Plasma



Evolution of the nuclear medium as seen through jets



- The nuclear fluid is created during pre-equilibrium dynamics stage, where most of the collision's $T^{\mu\nu}$ will be in the fluid.
- Hydrodynamical stage (Temp $\sim 10^2$ MeV): Strongly coupled quark gluon plasma (QGP)
  - Equation of State (EoS) computed via Lattice QCD
- Molecular dynamics stage (Temp ~ 10 MeV): $\lambda_{micro} \sim L_{hydro}$, simulation switches to Boltzmann transport
- Following free-streaming, soft hadrons ($p_T \lesssim$ 3 GeV/c) carry most of the medium's $T^{\mu\nu}$ to detectors.

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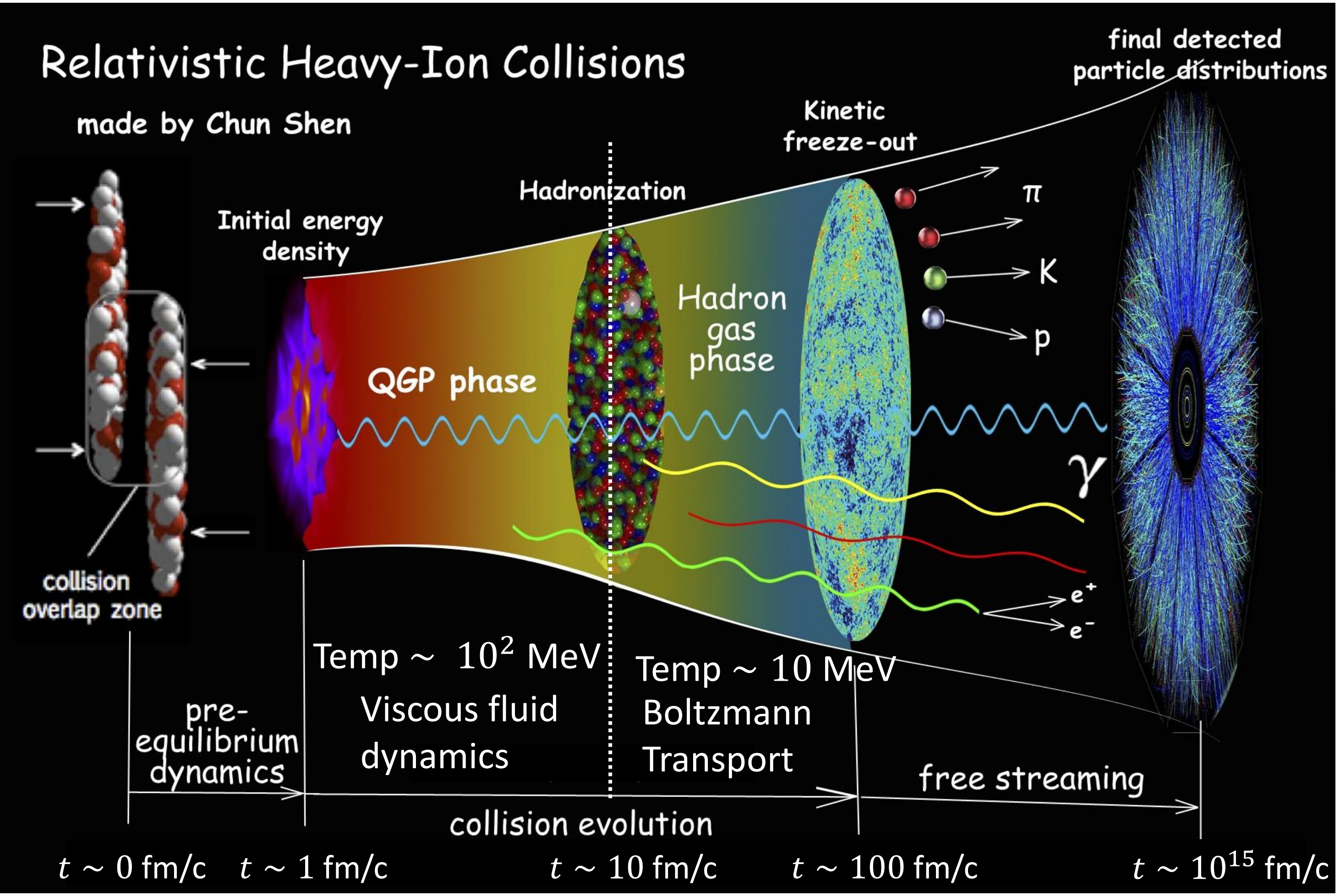

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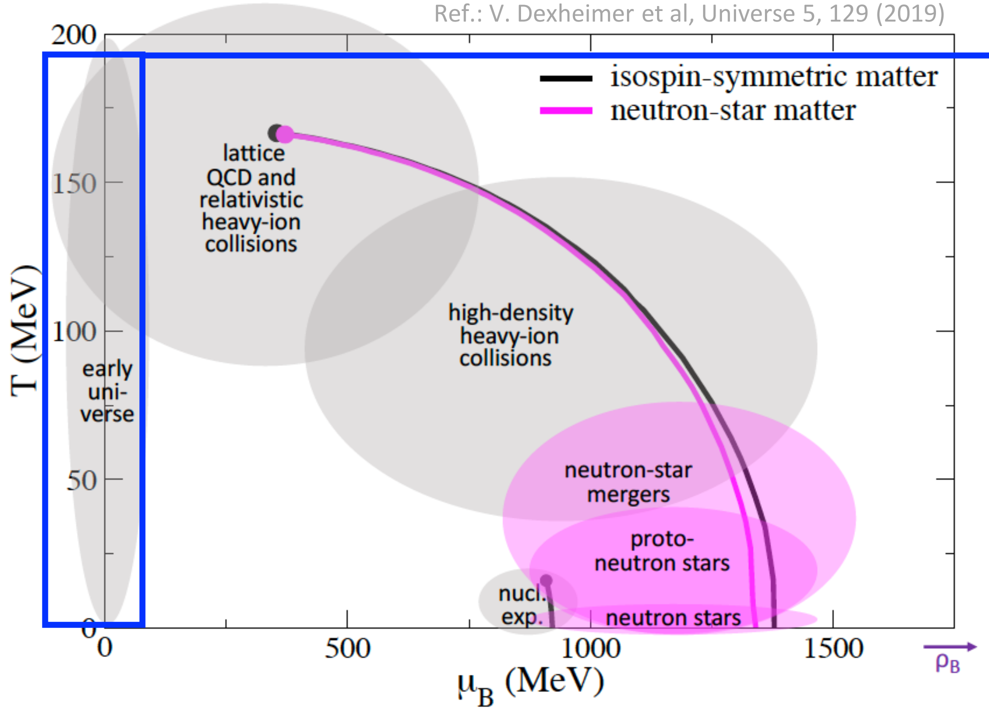
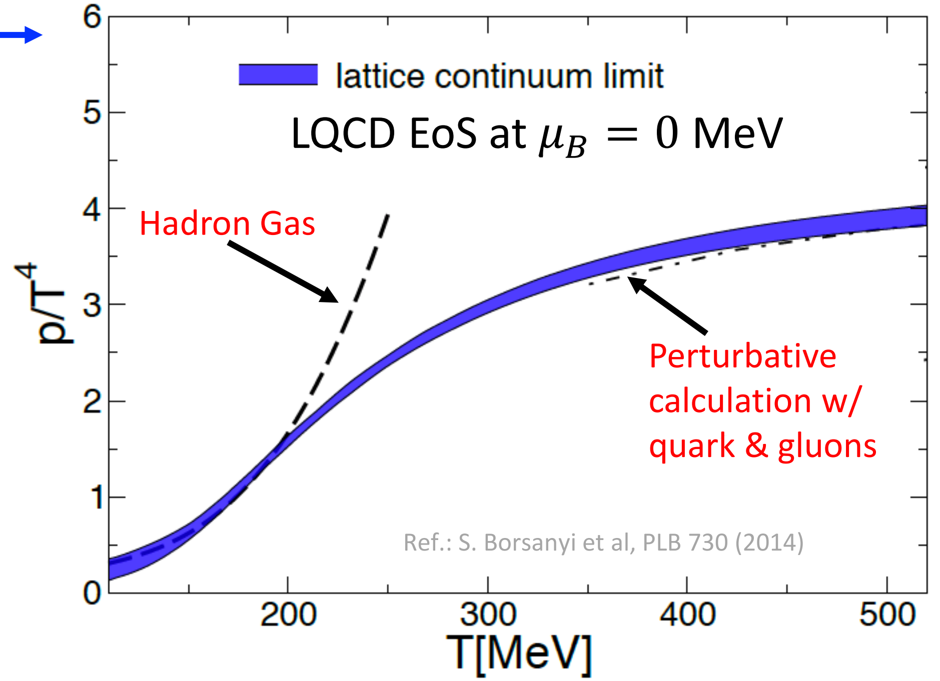
- To help simulate these different aspects of heavy-ion collisions, the JETSCAPE (Jet Energy-loss Tomography with a Statistically and Computationally Advanced Program Envelope) framework was used.

Nuclear equation of state in thermal equilibrium

- $\uparrow \sqrt{s_{NN}} \Rightarrow$ more gluons $\Rightarrow n_q \sim n_{\bar{q}} \Rightarrow \mu_B \approx 0$
- $\downarrow \sqrt{s_{NN}} \Rightarrow$ more valance quarks $\Rightarrow \mu_B > 0$

Lattice QCD (L-QCD) equation of state (EoS)





μ_B : (net) Baryon chemical potential

$\mu_B \propto (number\ of\ baryons) - (number\ antibaryons)$

Overview of fluid dynamics

- $P(T)$ can be used to describe fluids in perfect thermal equilibrium
- Is a perfect thermal equilibrium created after a nucleus-nucleus collision?
  - Extensive studies revealed that the QGP fluid is out-of-equilibrium, i.e. it’s dissipative or viscous

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- Fluid perturbations and dissipation:

$$Pert. \propto \exp\left[i\underbrace{(c_s kt - \vec{k}\cdot\vec{x})}^{\text{Wave propagation of perturbations at speed of sound } c_s} - \underbrace{\frac{4\eta}{3s}\frac{k}{2T}kt}_{\text{Decay/Dissipation of perturbations}} \right]$$

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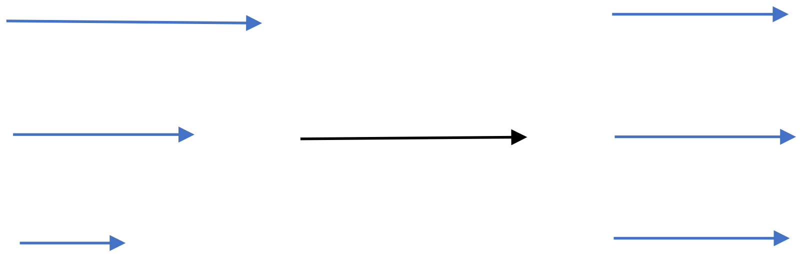
- Specific shear viscosity η/s is a transport coefficient
  - η shear viscosity
  - s entropy density
- η introduces friction between fluid layers



An irreducible tensor decomposition of hydrodynamics

- In high-energy collisions (w/ negligible μ_B), what is flowing?... That can only be energy density ϵ, mass density is inappropriate: pair production & annihilation!

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$$T^{\mu\nu}u_\nu = \epsilon u^\mu$$

Landau's definition of fluid flow

$u^\mu = \left(\gamma, \gamma\vec{\beta}\right)$ where

$\gamma = \left(1-\beta^2\right)^{-1/2}$ and $\vec{\beta} = \vec{v}/c$. Using natural units from now on $\Rightarrow c = 1 = \hbar = k_B$

- Non-dissipative $T_0^{\mu\nu}$ can only take the form:

$$T_0^{\mu\nu} = \epsilon u^\mu u^\nu - P(\epsilon)\Delta^{\mu\nu} = \epsilon u^\mu u^\nu - P(\epsilon)(g^{\mu\nu} - u^\mu u^\nu)$$

Note: $u^2 \equiv 1 \Rightarrow \dot{u}^\mu u_\mu = 0$

$\Rightarrow$ Pressure drives $\dot{u}^\mu \perp u^\mu \Rightarrow P(\epsilon)\Delta^{\mu\nu}$

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- Including dissipation gives rise to dissipative corrections $\delta T^{\mu\nu}$ to $T_0^{\mu\nu}$, namely Π and $\pi^{\mu\nu}$

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Bulk viscous pressure

Shear viscous pressure tensor

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where the viscous pressures are decomposed in terms of irreducible tensors, namely

radial deformations

$$\Pi = -\tfrac{1}{3}\Delta^{\mu\nu}T_{\mu\nu} - P(\epsilon)$$

angular deformations

$$\pi^{\mu\nu} = T^{\langle\mu\nu\rangle} = \Delta^{\mu\nu}_{\alpha\beta}\, T^{\alpha\beta} = \left[\frac{1}{2}\left(\Delta^\mu_\alpha\Delta^\nu_\beta + \Delta^\mu_\beta\Delta^\nu_\alpha\right) - \frac{1}{3}\Delta^{\mu\nu}\Delta_{\alpha\beta}\right]T^{\alpha\beta}$$

w/ $\pi^\mu_\mu = 0$ and $u_\mu\pi^{\mu\nu} = 0$

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w/ $\pi^\mu_\mu = 0$ and $u_\mu\pi^{\mu\nu} = 0$

- The EoM for Π and $\pi^{\mu\nu}$ are from the Boltzmann equation.

H. Grad moment expansion of the Boltzmann equation

- Expanding the f_p in the Boltzmann equation $p^\mu \partial_\mu f_p = C[f_p]$ w/ irreducible moments

$$f_p = f_{0p} + \delta f_p = \underbrace{f_{0p}}_{\text{thermal distribution}}\Big[1 + \overbrace{G\big(p^0, |\vec{p}|\big)}^{\text{“radial” dep.}} \otimes \underbrace{\phi_p}_{\text{“angular” dep.}}\Big]$$

[J. Phys. G: Nucl. Part. Phys. **41**, 124004 (2014)]

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$$= f_{0p}\left[1 + \left\{\overbrace{G_0(p^0, |\vec{p}|)}^{\text{monopole}} + \underbrace{G_1(p^0, |\vec{p}|)c_{\langle\mu\rangle}p^{\langle\mu\rangle}}_{\text{dipole}} + \overbrace{G_2(p^0, |\vec{p}|)c_{\langle\mu\nu\rangle}p^{\langle\mu}p^{\nu\rangle}}^{\text{quadrupole}} + \cdots\right\}\right]$$

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- While the EoM for monopole and quadrupole deformations use Boltzmann equation

$$\Pi = -\frac{\Delta_{\alpha\beta}}{3} \int \frac{d^3p}{(2\pi)^3 p^0} p^\alpha p^\beta \delta f_\Pi; \qquad \pi^{\mu\nu} = \Delta^{\mu\nu}_{\alpha\beta} \int \frac{d^3p}{(2\pi)^3 p^0} p^\alpha p^\beta \delta f_\pi$$

Relativistic dissipative hydrodynamics from Grad's expansion

- Relativistic dissipative hydrodynamics

$$\partial_\mu T^{\mu\nu} = 0 \qquad P(\varepsilon) \text{ use lattice QCD EoS}$$

$$T^{\mu\nu} = \varepsilon u^\mu u^\nu - [P(\varepsilon) + \Pi]\Delta^{\mu\nu} + \pi^{\mu\nu}$$

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- The expanding the Boltzmann equation $p^\mu \partial_\mu f_p = C[f_p]$ using $\delta f_{\Pi,\pi}$ up to rank-2 tensors gives EoM for Π and $\pi^{\mu\nu}$:

$$\tau_\Pi \dot{\Pi} + \Pi = -\zeta\theta - \delta_{\Pi\Pi}\Pi\theta + \lambda_{\Pi\pi}\pi^{\mu\nu}\sigma_{\mu\nu} + \cdots$$

$$\theta = \partial_\mu u^\mu$$

$$\sigma^{\mu\nu} = \partial^{\langle\mu}u^{\nu\rangle}$$

$$\tau_\pi \dot{\pi}^{\langle\mu\nu\rangle} + \pi^{\mu\nu} = 2\eta\sigma^{\mu\nu} - \delta_{\pi\pi}\pi^{\mu\nu}\theta + \phi_7\pi_\alpha^{\langle\mu}\pi^{\nu\rangle\alpha} - \tau_{\pi\pi}\pi_\alpha^{\langle\mu}\sigma_\alpha^{\nu\rangle} + \lambda_{\pi\Pi}\Pi\sigma^{\mu\nu} + \cdots$$

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- The goal is to constrain ζ and η via Bayesian analysis [all transport coefficient are set c.f. PRD **85** 114047 (2012), PRC **90** 024912 (2014)]

Relativistic dissipative hydrodynamics

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$$p^0 \frac{d^3N}{d^3p} = \frac{1}{(2\pi)^3} \int d^3\Sigma_\mu p^\mu (f_0 + \delta f_\Pi + \delta f_\pi) \quad f_0 = g[\exp(E/T) + a]^{-1}$$

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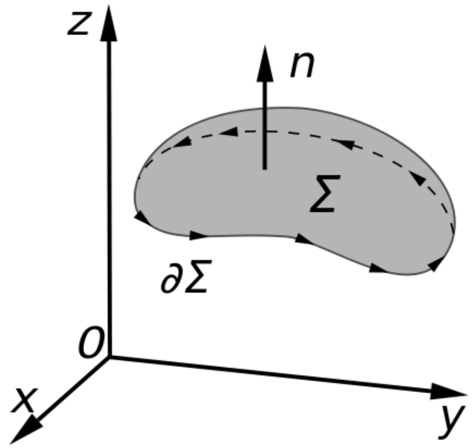

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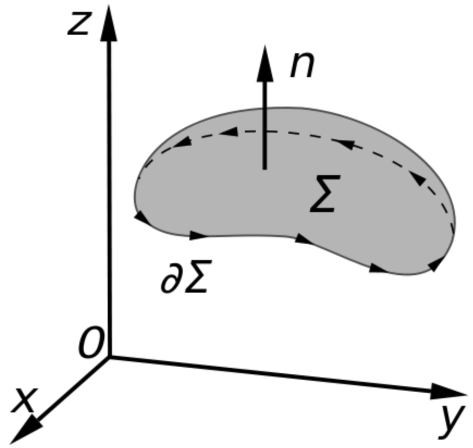

https://en.wikipedia.org/wiki/Stokes%27_theorem

- Grad's expansion, using $\delta f / f_0 < 1$, yields

$$\delta f_\pi = f_0(1 + af_0) A_\pi p_\mu p_\nu \pi^{\mu\nu} = f_0(1 + af_0) \frac{p_\mu p_\nu \pi^{\mu\nu}}{2(\epsilon + P)T^2}$$

$$\delta f_\Pi = f_0(1 + af_0)\Pi\left(A_E (p \cdot u)^2 + A_T m^2\right)$$

$$A_E, A_T \propto \int \frac{d^3p}{(2\pi)^3 p^0} (p \cdot u)^m (-p \cdot \Delta \cdot p)^n f_0 (1 + af_0)$$ thermodynamical integrals

[PRC **103**, 064903 (2021)]

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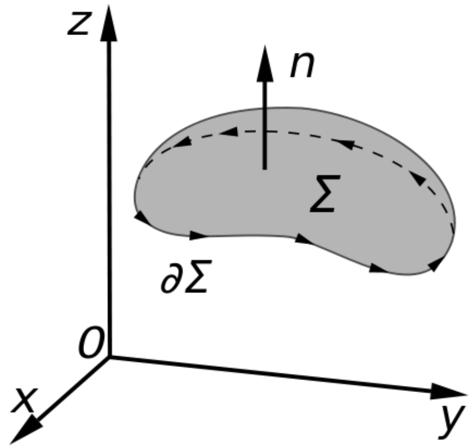

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- An alternative to the Grad expansion, Chapman-Enskog expansion uses small gradients (i.e. flow, μ_B, …) as expansion parameter ($\delta f / f_0 < 1$)

$$\delta f_\pi = f_0(1 + af_0) \frac{p_\mu p_\nu \pi^{\mu\nu}}{2(p \cdot u) J_{32}};$$

$$J_{rq} \propto \int \frac{d^3p}{(2\pi)^3 p^0} (p \cdot u)^{r-2q} (-p \cdot \Delta \cdot p)^q f_0(1 + af_0)$$

[PRC **103**, 064903 (2021)]

$$\delta f_\Pi = f_0(1 + af_0) \frac{\Pi}{\beta_\Pi} \left[\frac{(p \cdot u)\mathcal{F}}{T^2} - \frac{p \cdot \Delta \cdot p}{3T(p \cdot u)} \right];$$

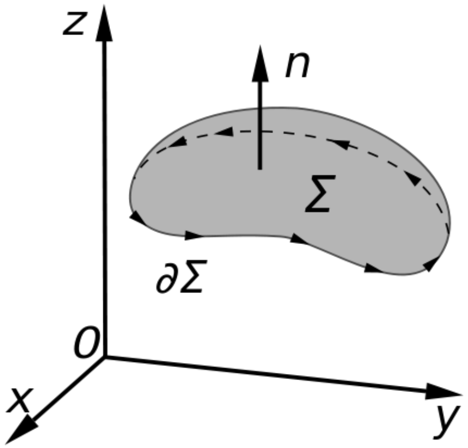
$$\beta_\Pi, \mathcal{F} \propto \int \frac{d^3p}{(2\pi)^3 p^0} (p \cdot u)^m (-p \cdot \Delta \cdot p)^n f_0(1 + af_0)$$

Relativistic dissipative hydrodynamics

- As $\epsilon \downarrow \Rightarrow \lambda_{mfp} \uparrow$. Once $\lambda_{mfp} \sim L$, the hydrodynamical approximation breaks down $\Rightarrow$ the full Boltzmann equation must be solved.

- Converting fluid degrees of freedom to particle distributions via the Cooper-Frye prescription

$$p^0 \frac{d^3N}{d^3p} = \frac{1}{(2\pi)^3} \int d^3\Sigma_\mu p^\mu (f_0 + \delta f_\Pi + \delta f_\pi) \quad f_0 = g[\exp(E/T) + a]^{-1}$$



https://en.wikipedia.org/wiki/Stokes%27_theorem

- An alternative to the Grad expansion, Chapman-Enskog expansion uses small gradients (i.e. flow, μ_B, …) as expansion parameter

$$\delta f_\pi = f_0(1 + af_0) \frac{p_\mu p_\nu \pi^{\mu\nu}}{2(p\cdot u) J_{32}};$$

$$J_{rq} \propto \int \frac{d^3p}{(2\pi)^3 p^0} (p\cdot u)^{r-2q} (-p\cdot\Delta\cdot p)^q f_0(1 + af_0)$$

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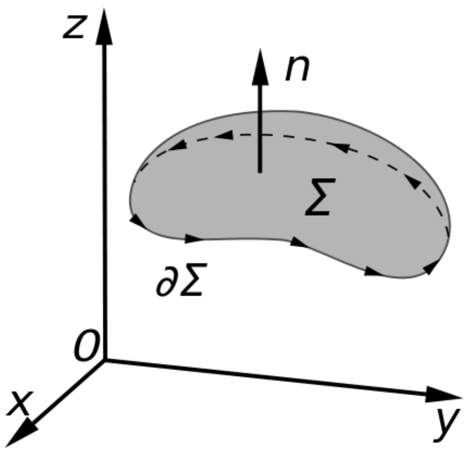
- Note that Chapman-Enskog expansion gives the same equations of motion for Π and $\pi^{\mu\nu}$ as Grad's moments, however with different transport coefficients. [PRC **89**, 054903 (2014)]

Relativistic dissipative hydrodynamics

- Converting fluid degrees of freedom to particle distributions via the Cooper-Frye prescription

$$p^0 \frac{d^3N}{d^3p} = \frac{1}{(2\pi)^3} \int d^3\Sigma_\mu p^\mu f$$



https://en.wikipedia.org/wiki/Stokes%27_theorem

- If $\frac{\delta f}{f_0} \sim 1$ is present, a resummed expansion follows the *ansatz* suggested by Pratt-Torrieri-Bernhard [PRC **103**, 064903 (2021)]

$$f = \frac{\mathcal{Z}_\Pi}{\det(\Lambda)} \left[\exp\left(\frac{\sqrt{|\vec{p}'|^2 + m^2}}{T} \right) + a \right]^{-1} \quad a = \begin{cases} 1 & \text{Bose} - \text{Einstein} \\ 0 & \text{Boltzmann} \\ -1 & \text{Fermi} - \text{Dirac} \end{cases}$$

$$\mathcal{Z}_\Pi = \frac{\Pi + P(\varepsilon)}{L_{21}} \qquad \vec{p}' = \Lambda^{-1}\vec{p}$$

$$\Lambda_{ij} = (1 + \lambda_\Pi)\delta_{ij} + \frac{\pi_{ij} T}{2J_{32}}$$

$$J_{rq} \propto \int \frac{d^3p}{(2\pi)^3 p^0} (p \cdot u)^{r-2q} (-p \cdot \Delta \cdot p)^q f_0 (1 + a f_0)$$

$$L_{rq} \propto \int \frac{d^3p}{(2\pi)^3 p^0} (p \cdot u)^{r-2q} (-p \cdot \Delta \cdot p)^q f$$

Relativistic dissipative hydrodynamics

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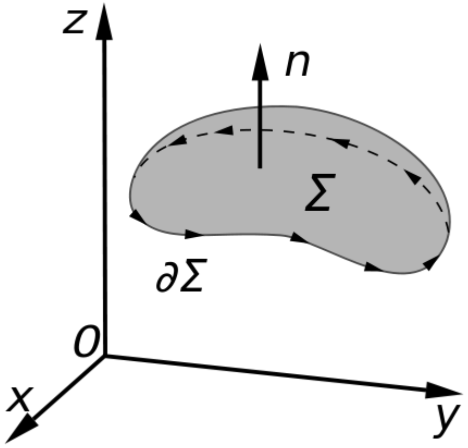


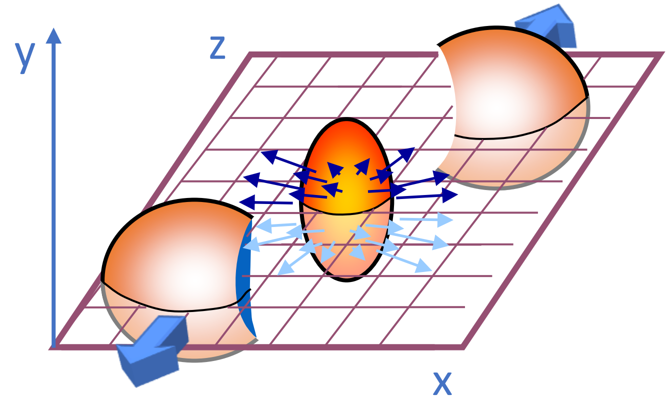
https://en.wikipedia.org/wiki/Stokes%27_theorem

- The approximations for f are :
  - Grad moment approximation (up to 2nd moment) linearizes $f \to f_0 + \delta f$
  - Chapman-Enskog (small) gradient approximation linearizes $f \to f_0 + \delta f$
  - Pratt-Torrieri-Bernhard deformed (thermal-like) distribution (non-linear f)

- The goal is to investigate the constraints on the shear (η) and bulk (ζ) viscosity from measurements of $p^0 \frac{d^3N}{d^3p}$ using various hadrons & contrast various f results.

A measure of anisotropic flow (v_n)

- Elliptic Flow



- A nucleus-nucleus collision is typically not head on;
an almond-shape region of matter is created.
- To quantify this almond-shape region, the centrality is introduced, where 0-10% being the 10% most head-on collisions, while 40-50% being semi-peripheral collisions shown.

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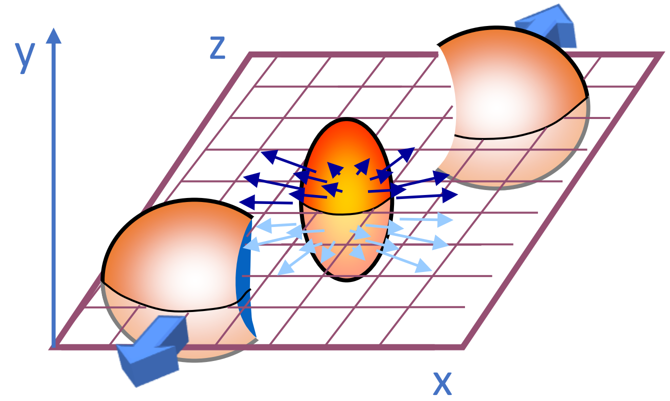

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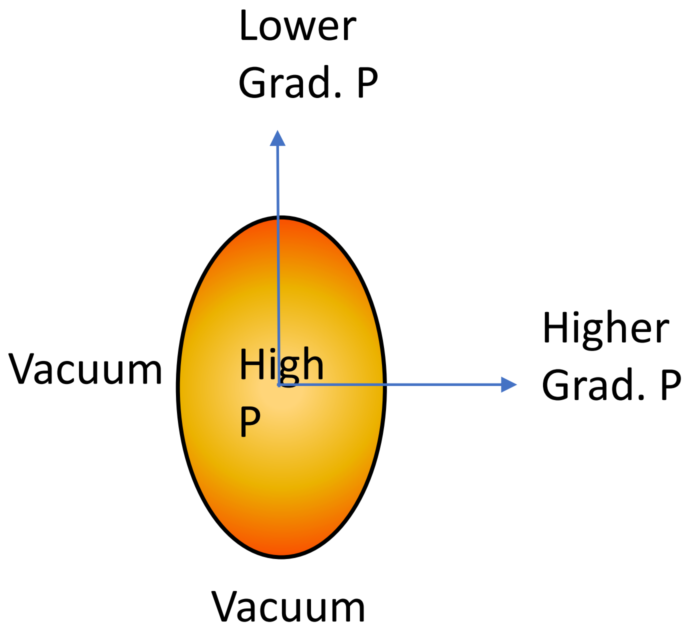

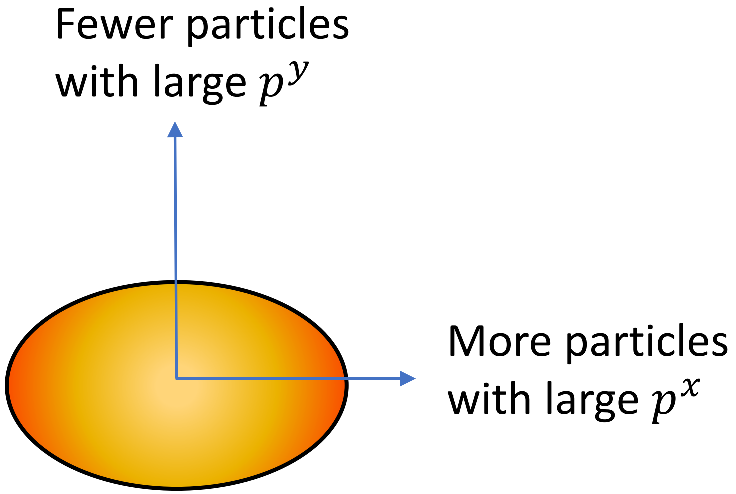

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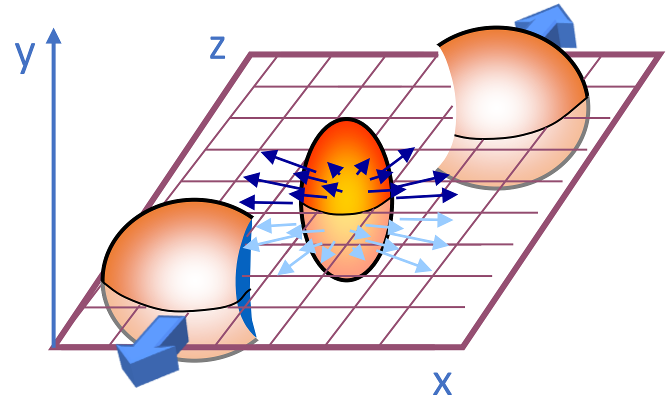

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- To describe the angular (ϕ) momentum distribution (in x-y plane, i.e. $\vec{p}_{\perp}$), use a Fourier decomposition (i.e. flow coefficients) v_n

$$\frac{dN}{dM d\eta_p p_\perp dp_\perp d\phi} = \frac{1}{2\pi} \frac{dN}{dM d\eta_p p_\perp dp_\perp} \left[1 + \sum_n v_n \cos(n\phi) \right]$$

$$\eta_p = \frac{1}{2} \mathrm{Log}\left[\frac{E_p + p^z}{E_p - p^z}\right]$$

- Second Fourier coefficient: elliptic flow (v_2) is the largest.

A measure of anisotropic flow (v_n)

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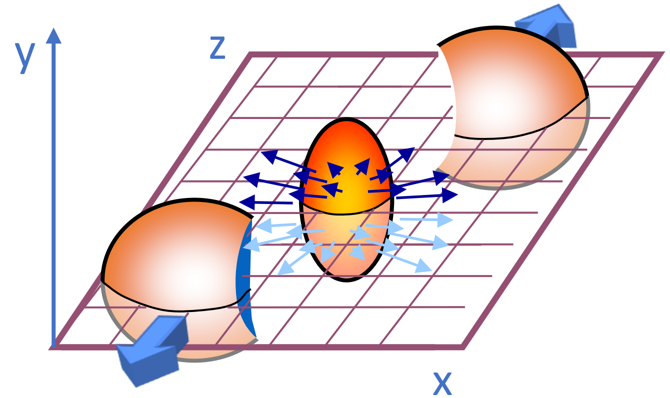

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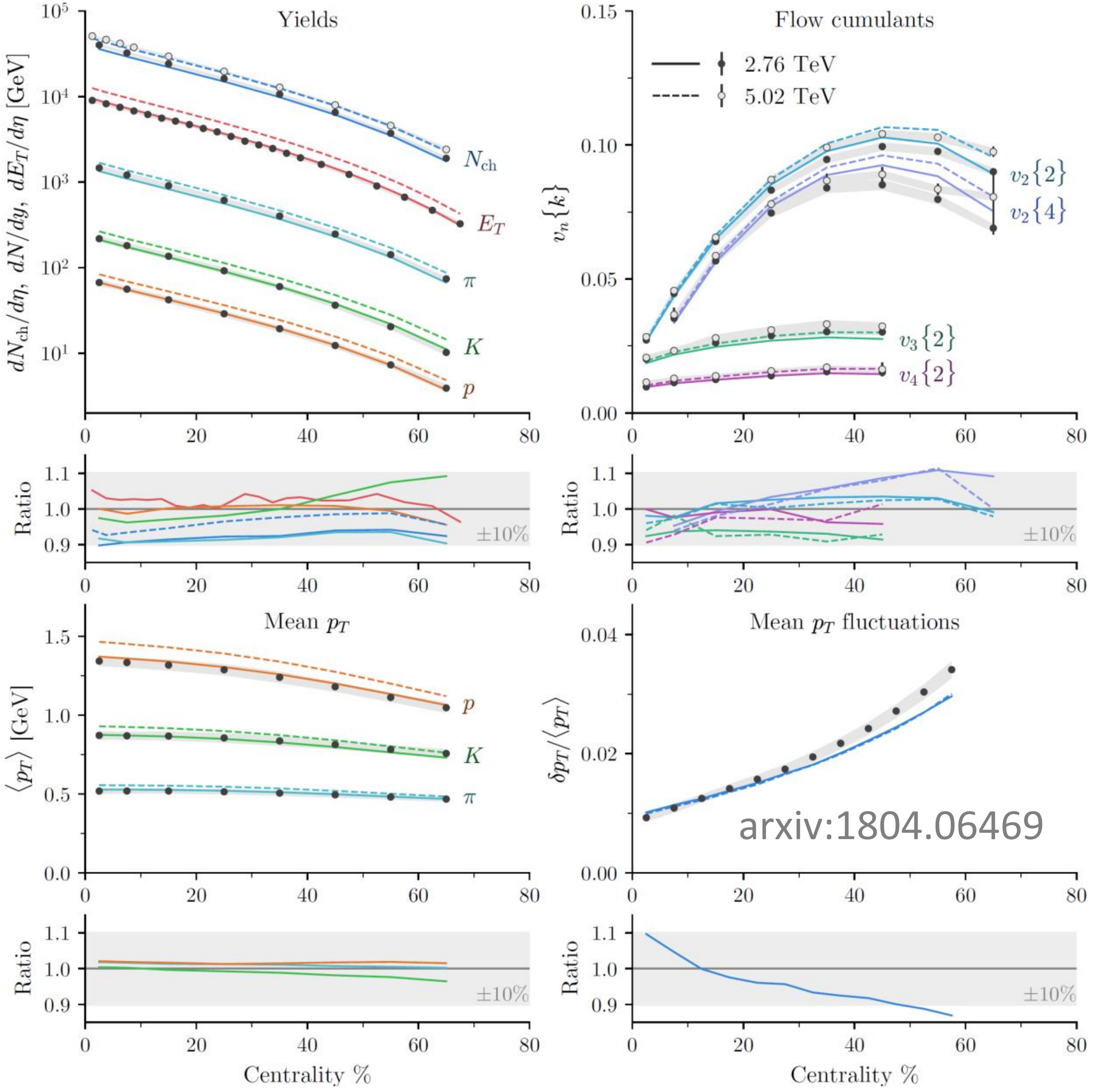
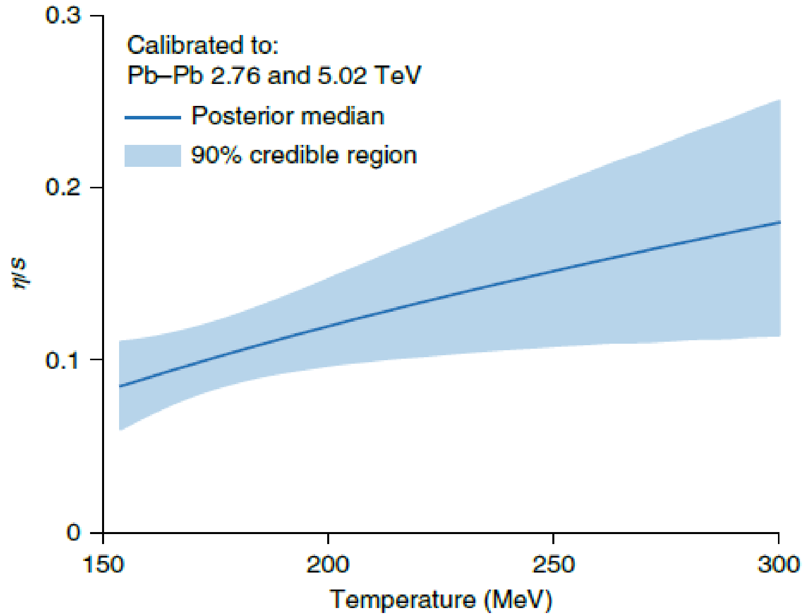
- Second Fourier coefficient: elliptic flow (v_2) is the largest.
- The more circular the $\vec{p}_T$-distribution $\Rightarrow$ smaller v_2, while the more elliptical $\Rightarrow$ larger the v_2.

A recent Bayesian analysis constraining $\frac{\eta}{s}$ and $\frac{\zeta}{s}$

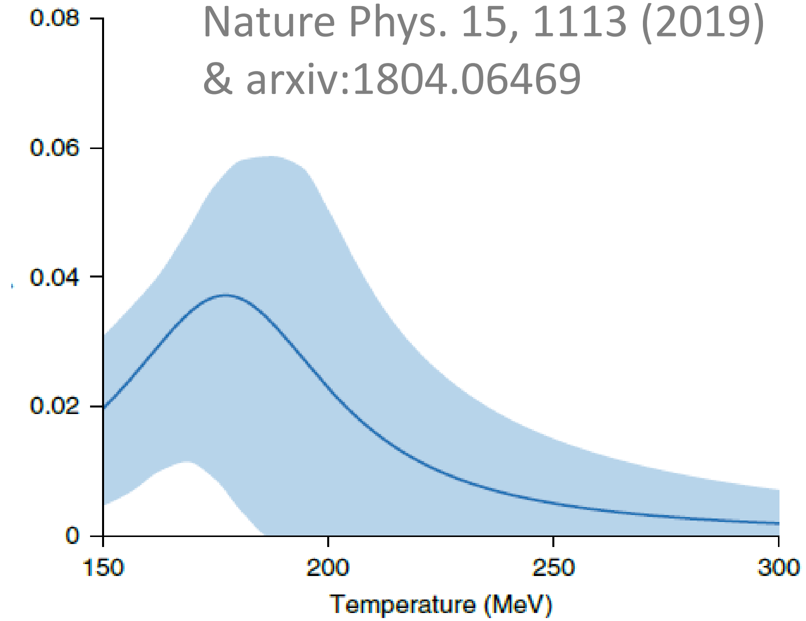


- The Bayesian analysis constrains parameters in the fluid simulation using various LHC Pb-Pb data ($\sqrt{s_{NN}} = 2.76\ TeV$ & $\sqrt{s_{NN}} = 5.02\ TeV$):
  - Multiplicity of identified particles
  - Average p_T of identified particles
  - Anisotropic flow v_n
  - Fluctuations around $\langle p_T \rangle$
- Constraints using the resummed *ansatz* of Pratt-Torrieri-Bernhard for δf

A recent Bayesian analysis constraining $\frac{\eta}{s}$ and $\frac{\zeta}{s}$



Nature Phys. 15, 1113 (2019) & arxiv:1804.06469



- Constraints using the Pratt-Torrieri-Bernhard δf *ansatz*

$$\frac{\eta}{s}(T) = \left(\frac{\eta}{s}\right)_{\min} + \left(\frac{\eta}{s}\right)_{\text{slope}} (T - T_c)\left(\frac{T}{T_c}\right)^{\left(\frac{\eta}{s}\right) crv} \Theta(T - T_c)$$

$$T_c = 0.154\, GeV$$

$$\frac{\zeta}{s}(T) = \frac{\left(\frac{\zeta}{s}\right)_{\max} \left(\frac{\zeta}{s}\right)^2_{width}}{\left(\frac{\zeta}{s}\right)^2_{widht} + (T - T_0)^2}$$

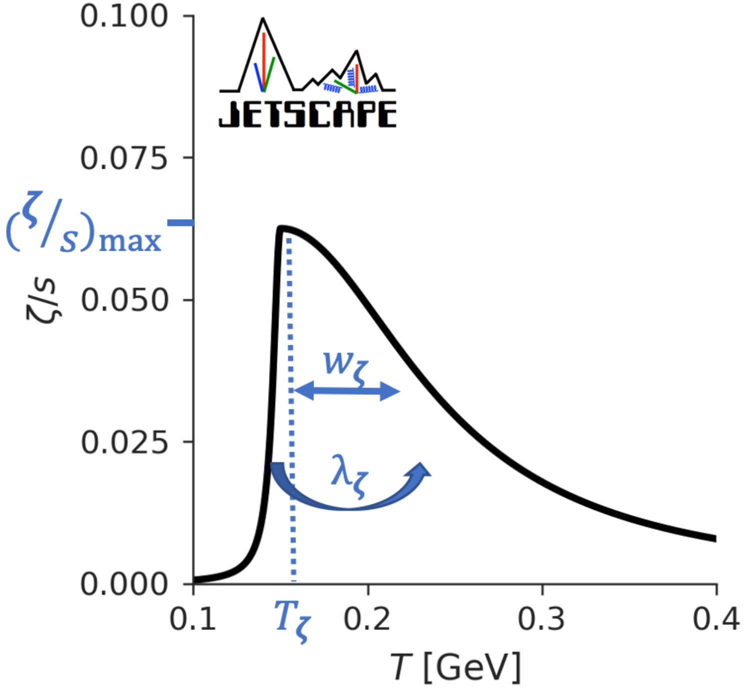
Initial condition / Pre-eq		QGP medium	
Norm	$13.9^{+1.2}_{-1.1}$ (2.76 TeV)	η/s min	$0.085^{+0.026}_{-0.025}$
	$18.5^{+1.8}_{-1.7}$ (5.02 TeV)	η/s slope	$0.83^{+0.83}_{-0.83}$ GeV^{-1}
p	$0.006^{+0.078}_{-0.078}$	η/s crv	$-0.37^{+0.79}_{-0.63}$
σ_{fluct}	$0.90^{+0.24}_{-0.27}$	ζ/s max	$0.037^{+0.040}_{-0.022}$
w	$0.96^{+0.04}_{-0.05}$ fm	ζ/s width	$0.029^{+0.045}_{-0.026}$ GeV
$d_{\min}$	$1.28^{+0.42}_{-0.53}$ fm	ζ/s T_0	$0.177^{+0.023}_{-0.021}$ GeV
τ_{fs}	$1.16^{+0.29}_{-0.25}$ fm/c	T_{switch}	$0.152^{+0.003}_{-0.003}$ GeV

LHC @ $\sqrt{s_{NN}} = 2.76\, TeV$ & $\sqrt{s_{NN}} = 5.02\, TeV$

Modelling specific bulk (ζ/s) and shear (η/s) viscosities

- Bulk and shear viscosities were parametrized using 4-parameter functions

PRC **103** 054904 (2021)



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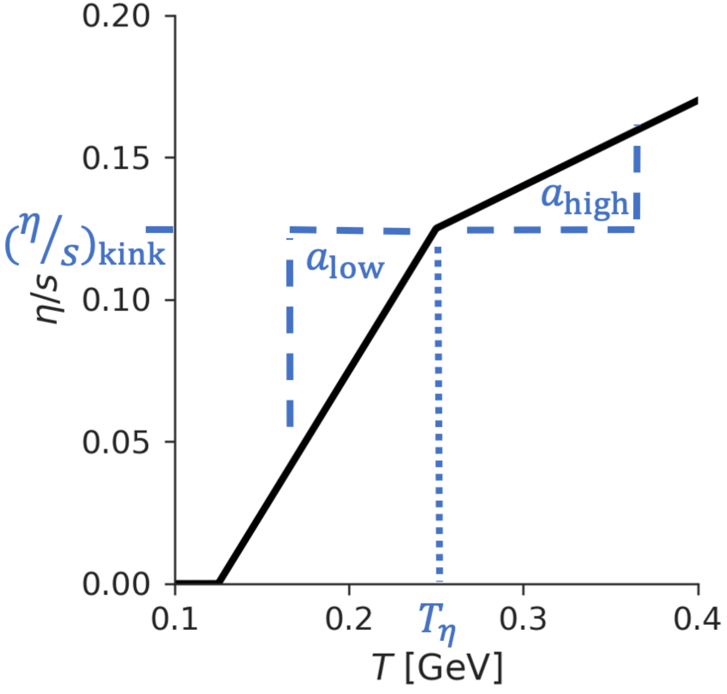
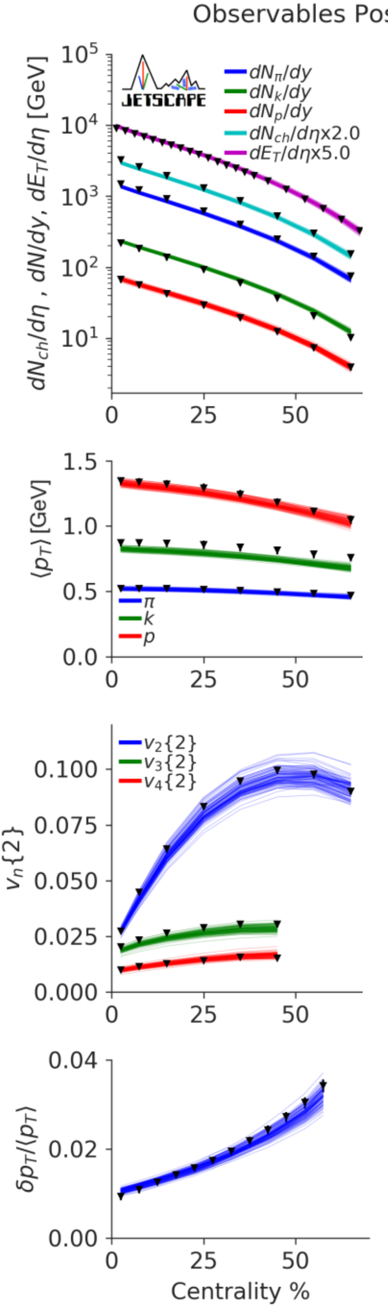


$$\frac{\zeta}{s}(T) = \frac{\left(\frac{\zeta}{s}\right)_{\max} \Lambda^2}{\Lambda^2 + \left(T - T_\zeta\right)^2}$$

$$\Lambda = w_\zeta\left[1 + \lambda_\zeta\left(T - T_\zeta\right)\right]$$

$$\frac{\eta}{s}(T) = a_{\mathrm{low}}\left(T - T_\eta\right)\Theta\left(T_\eta - T\right) + \left(\frac{\eta}{s}\right)_{\mathrm{kink}} + a_{\mathrm{high}}\left(T - T_\eta\right)\Theta\left(\mathrm{T} - \mathrm{T}_\eta\right)$$

Comparisons w/ experimental data using Bayesian calibration



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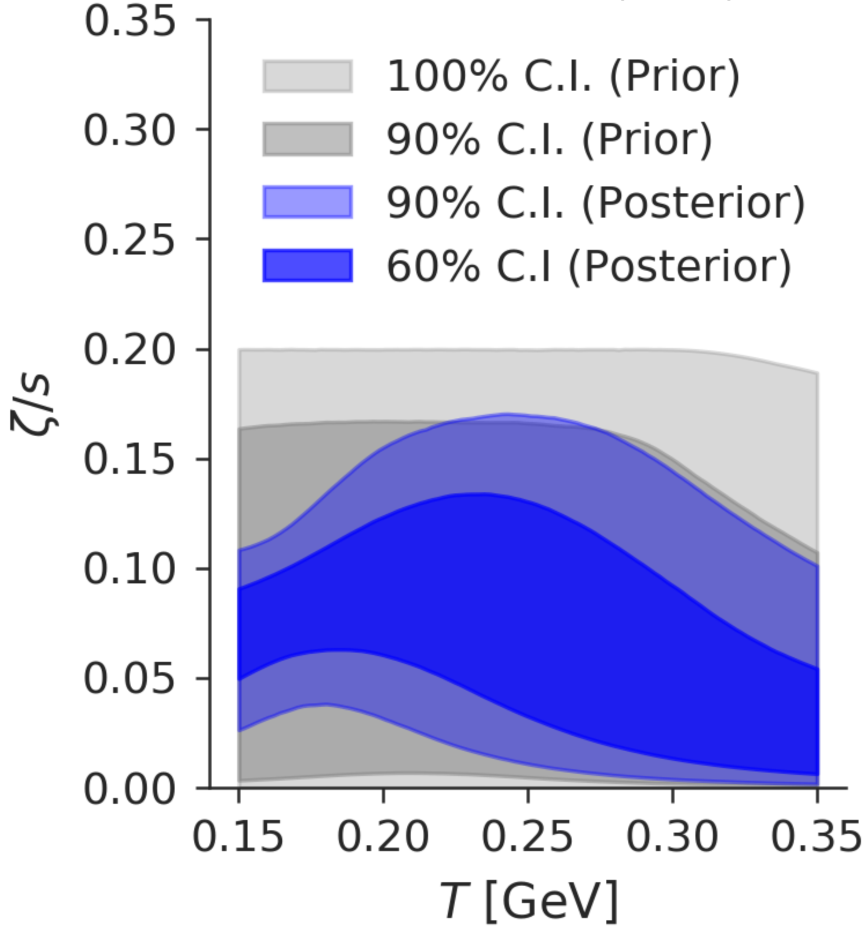

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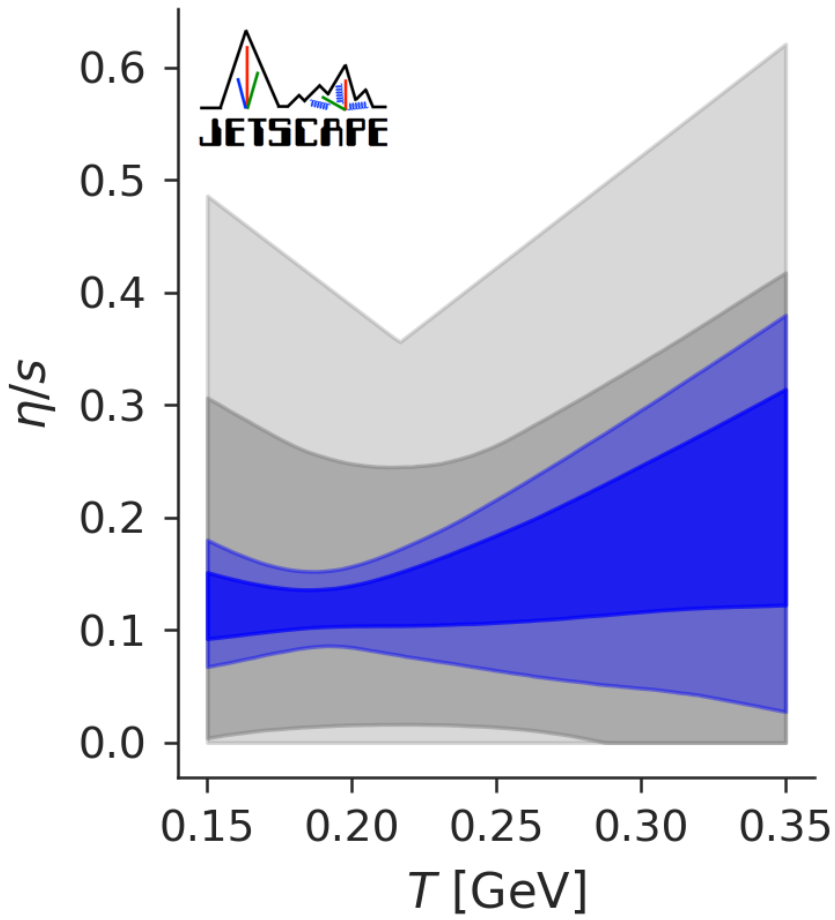
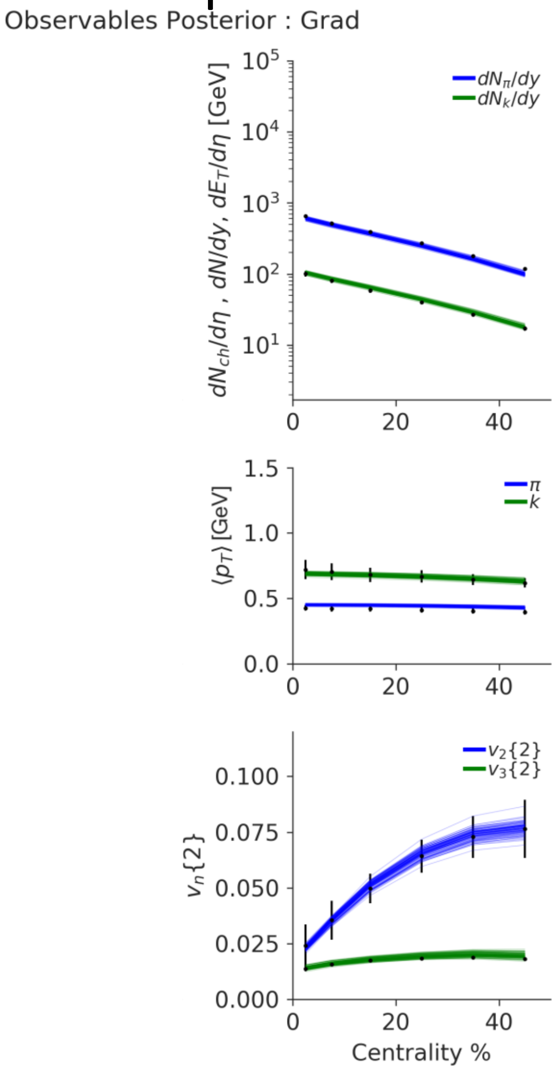


- Constraints on viscosities using only ALICE LHC @$\sqrt{s_{NN}}$ = 2.76 TeV data and Grad's δf

Comparisons w/ experimental data using Bayesian calibration



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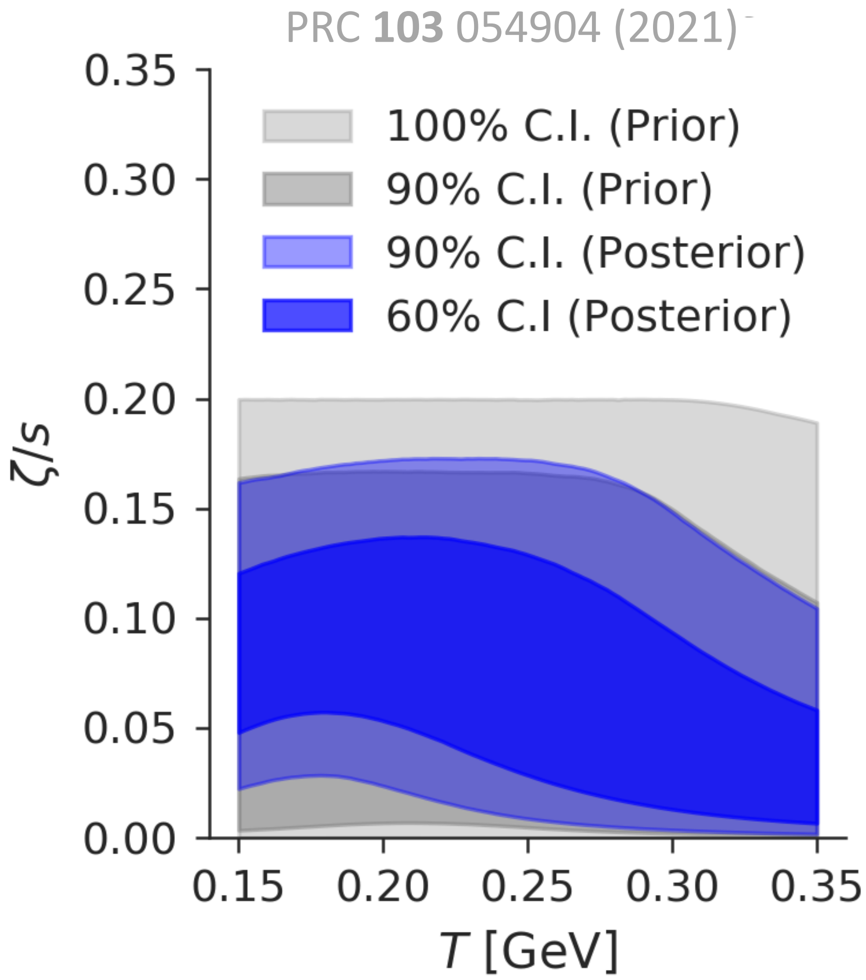
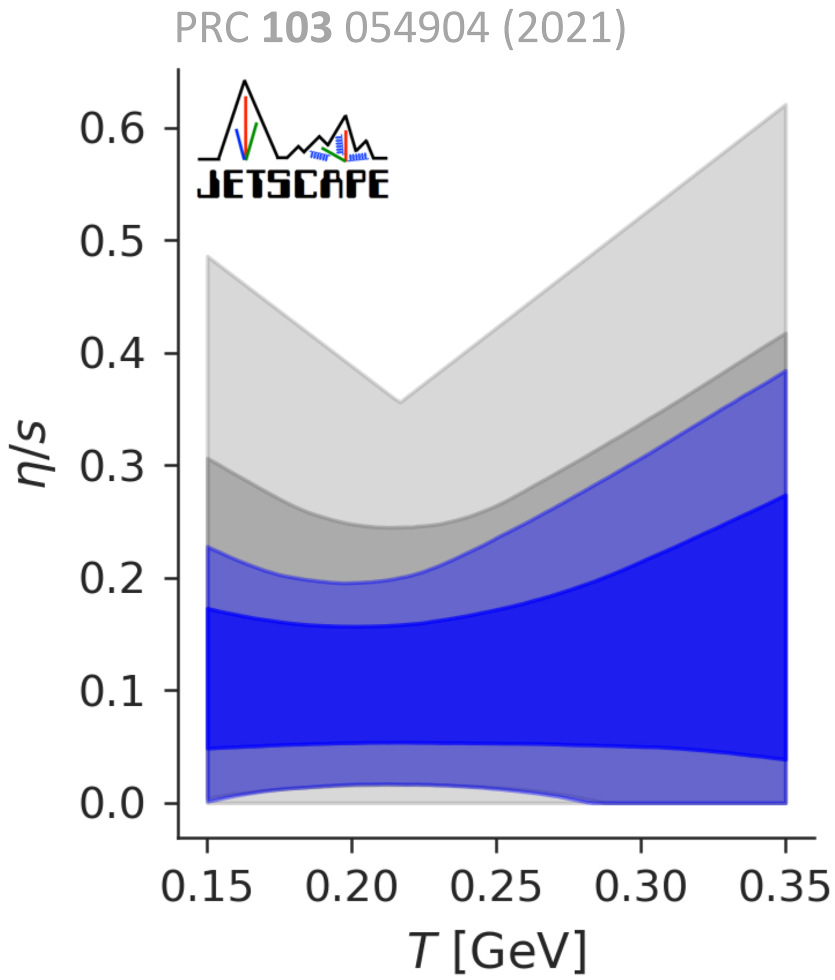
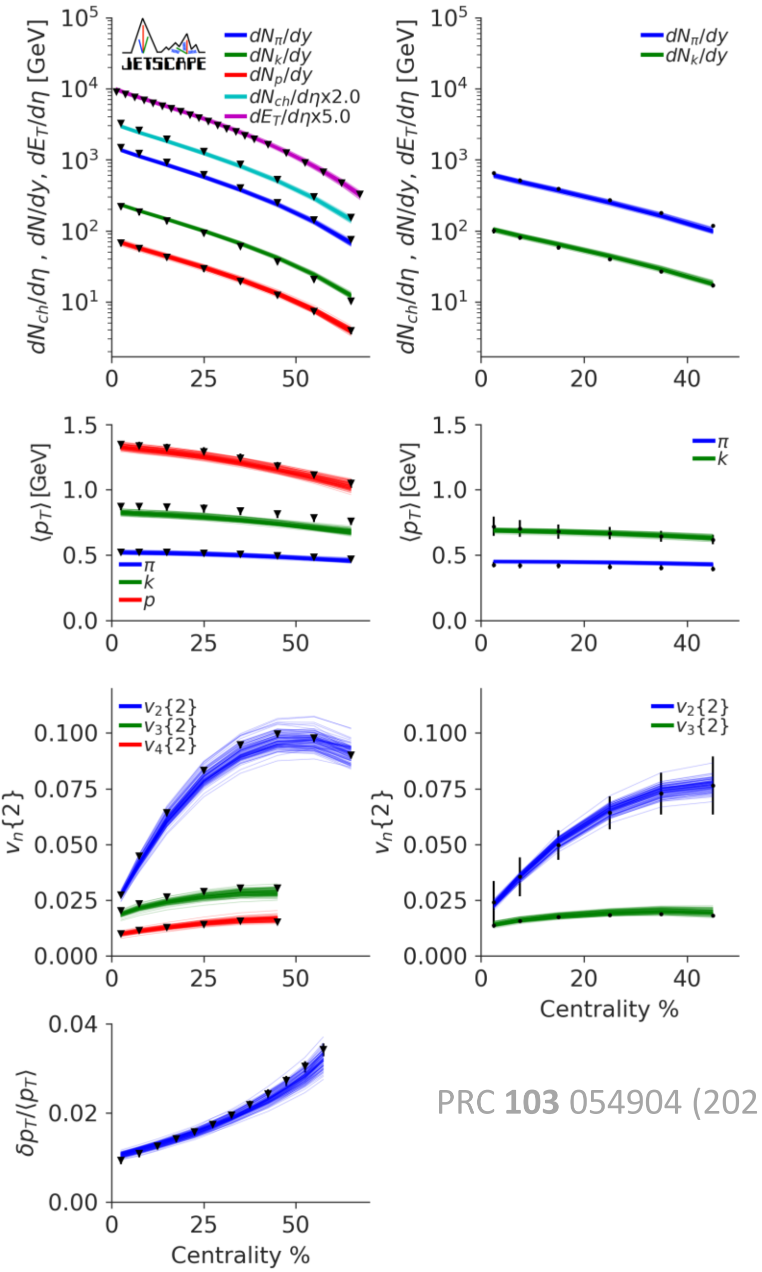




- Constraints on viscosities using only STAR RHIC @$\sqrt{s_{NN}} = 200\ GeV$ data and Grad's δf

Comparisons w/ experimental data using Bayesian calibration

Observables Posterior : Grad



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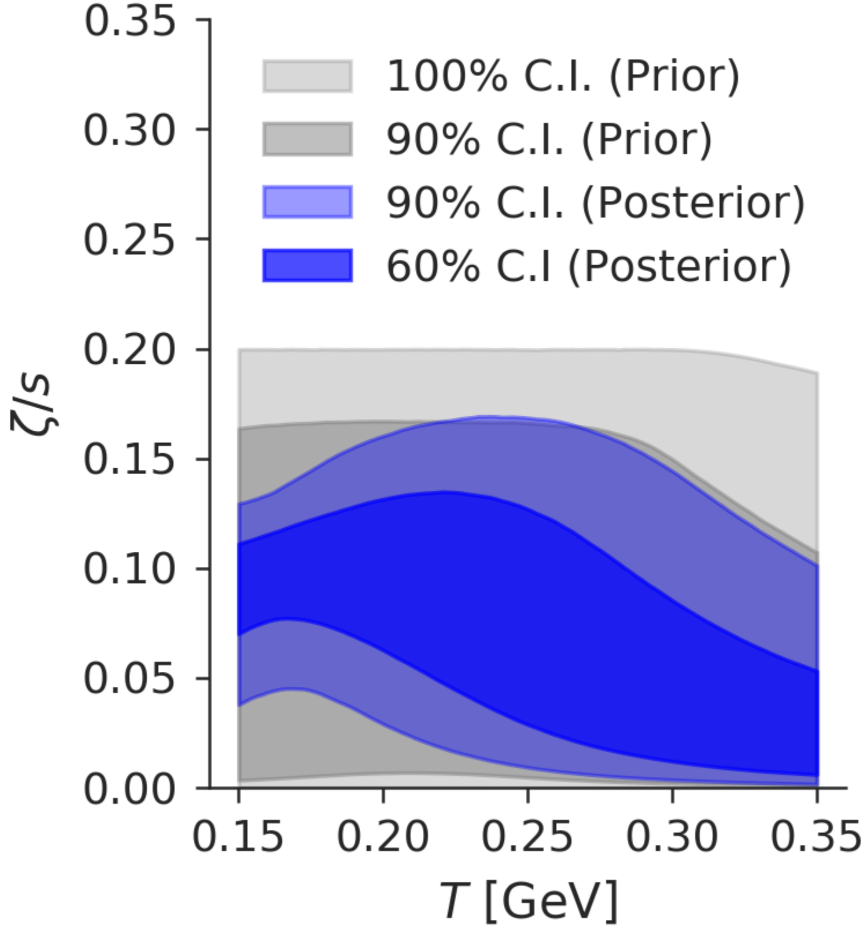

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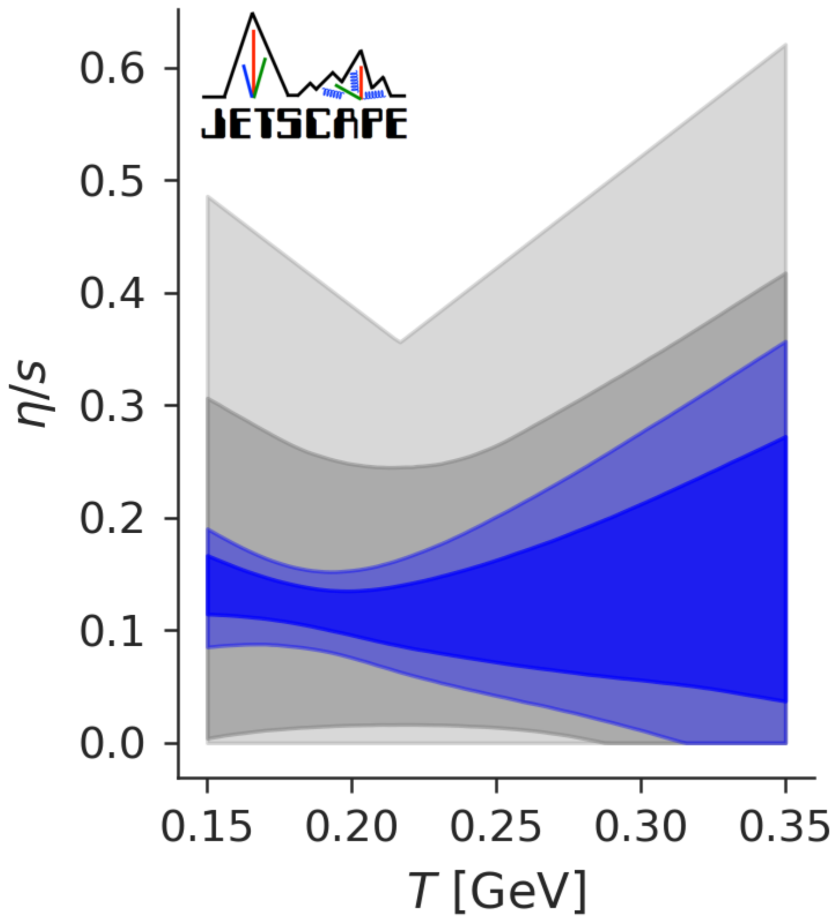
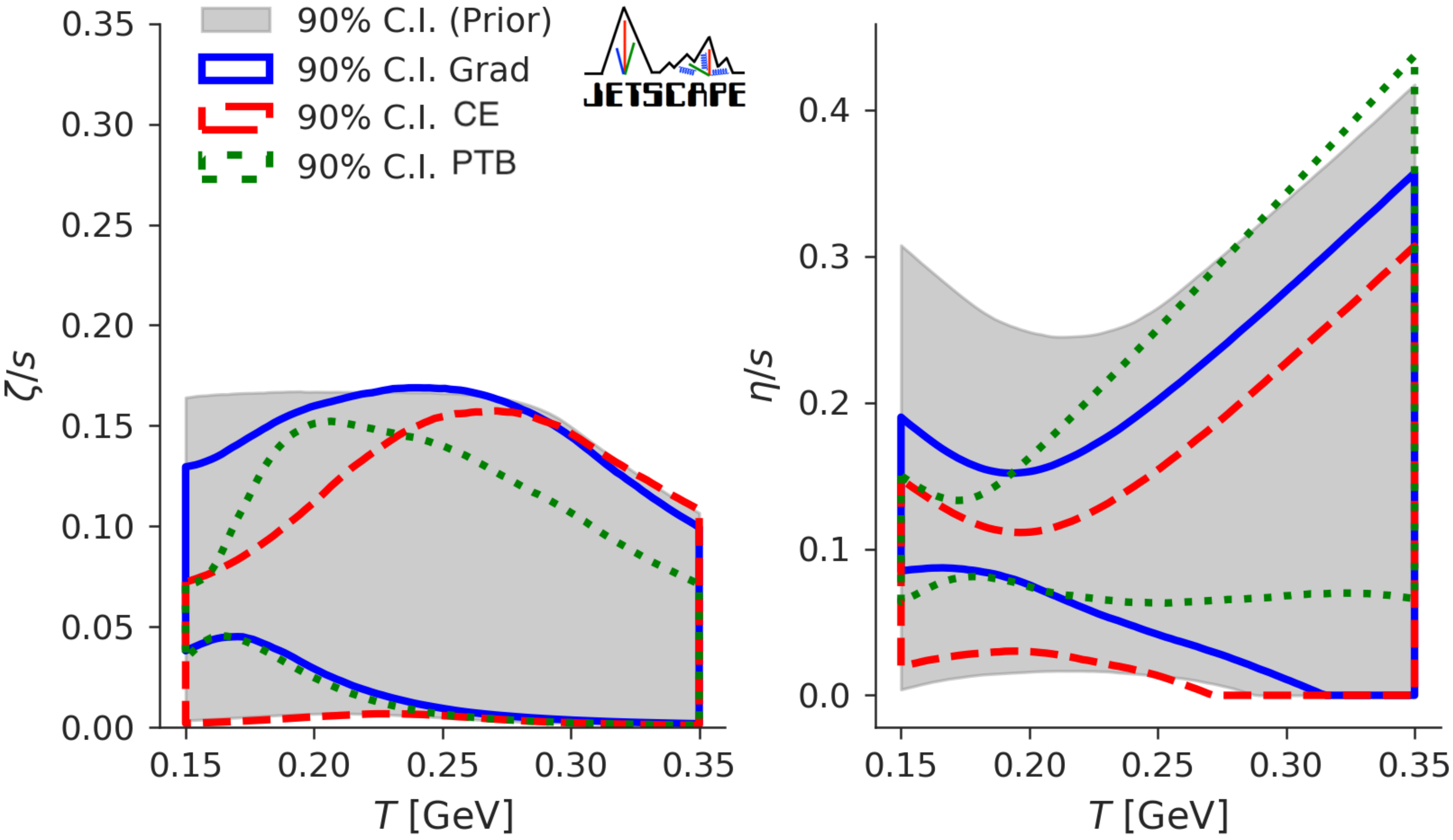


- Constraint on viscosities using RHIC and LHC data and Grad's δf

Combining different δf results using Bayesian Model Averaging



PRC **103** 054904 (2021)

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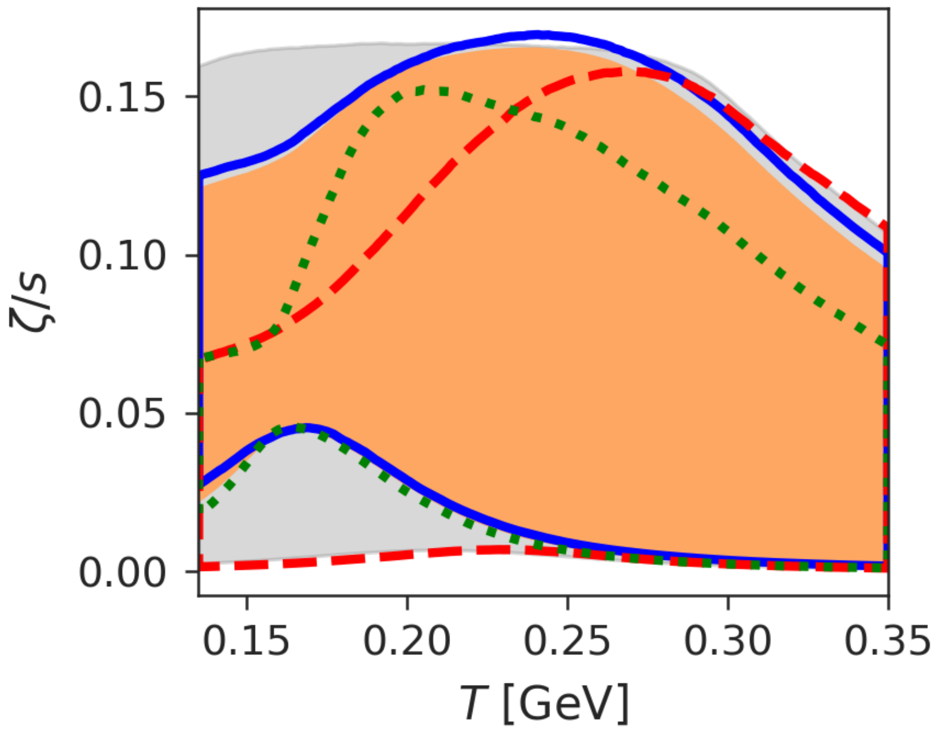
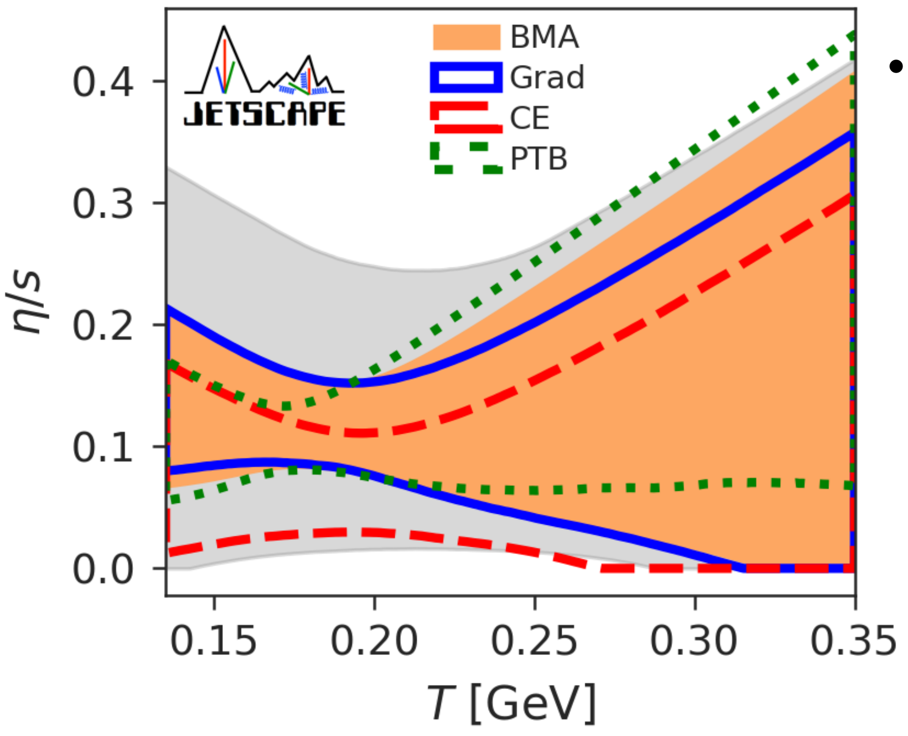
- The constraints on ζ/s and η/s from three different models
  - Grad's δf (blue)
  - Chapman-Enskog δf (red)
  - Pratt-Torrieri-Bernhard model (green)

Combining different δf results using Bayesian Model Averaging



PRL **126**, 242301 (2021)



PRL **126**, 242301 (2021)

- The constraints on ζ/s and η/s from three different models
  - Grad's δf (blue)
  - Chapman-Enskog δf (red)
  - Pratt-Torrieri-Bernhard model (green)

- Computing the Bayes factor (i.e. Bayesian evidence) allows to say that there are
  - 5000:1 odds that the Grad model is better than the Chapman-Enskog model, or 3.6σ observation.
  - 3:1 odds that the Grad model is better than the Pratt-Torrieri-Bernhard model, or a 0.6σ observation.
- Combining the three-models in proportion 5000:2000:1 using Bayesian Model Averaging (BMA), yields the robust constraints in orange. This is the first use of BMA in heavy-ion physics.

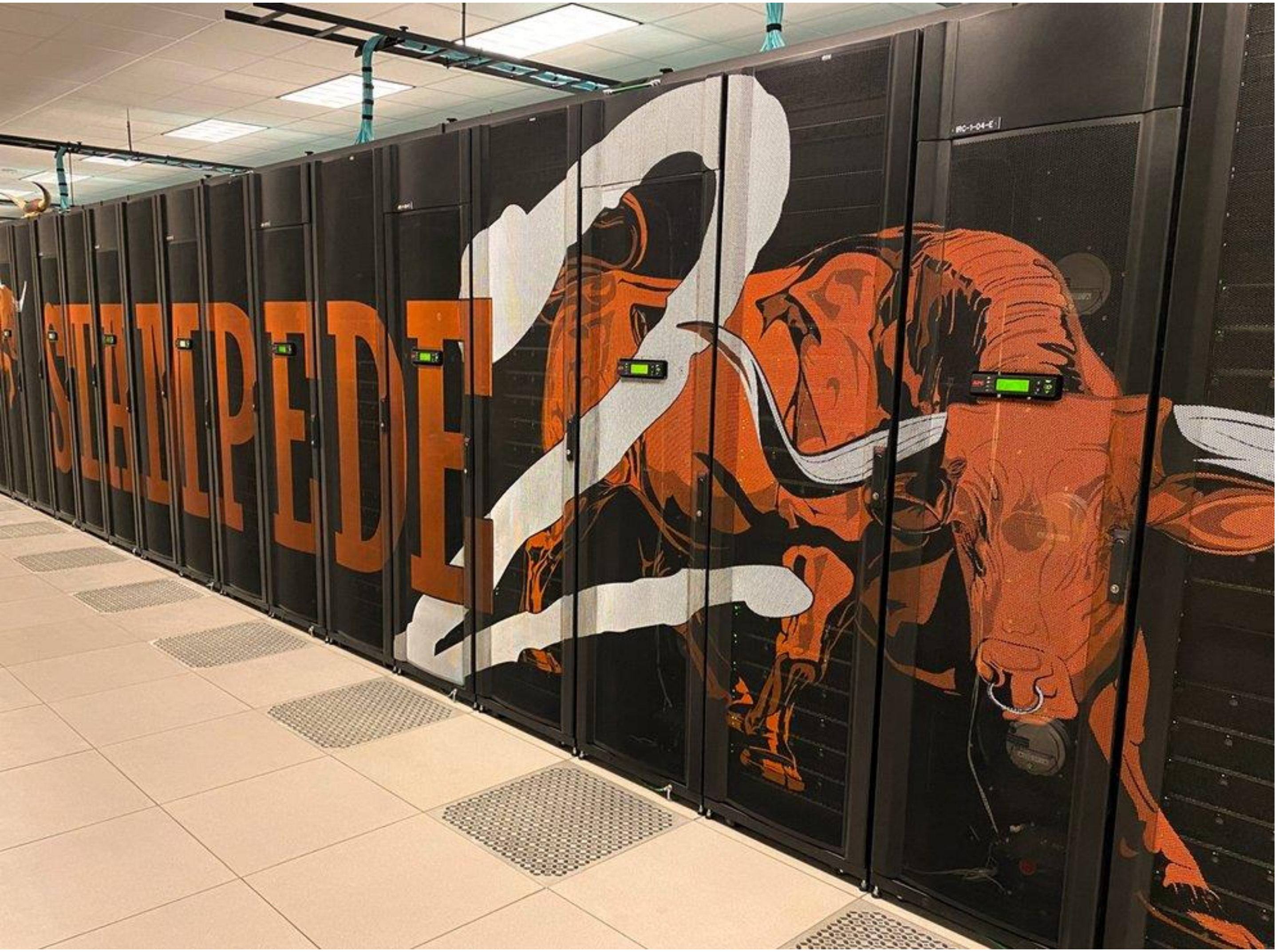
Conclusion and Outlook

- Modern simulations of heavy-ion collisions rely on a combination of relativistic dissipative fluid dynamics and far-off-equilibrium Boltzmann transport.
- Using hydrodynamics and Boltzmann transport, constraints on the QGP $\frac{\zeta}{s}(T)$ and $\frac{\eta}{s}(T)$ were obtained.
- These constraints are made more reliable by
  - Including multiple systems (RHIC and LHC)
  - Including an important theoretical systematic uncertainty δf along with Bayesian Model Averaging when extracting $\frac{\zeta}{s}(T)$ and $\frac{\eta}{s}(T)$.

- In the future, a more holistic Bayesian analysis using both hadrons as well as electromagnetic (EM) radiation will yield better constraints:
  - $\frac{\zeta}{s}(T)$ and $\frac{\eta}{s}(T)$ [PRC **93**, 044906 (2016); PRC **98**, 014902 (2018); PRC **101**, 044904 (2020)]
  - second order transport coefficients $(\mathrm{e.g.}\, \tau_\pi)$ and $\delta T^{\mu\nu}$ initial conditions [PRC **94**, 014904 (2016)]

Backup

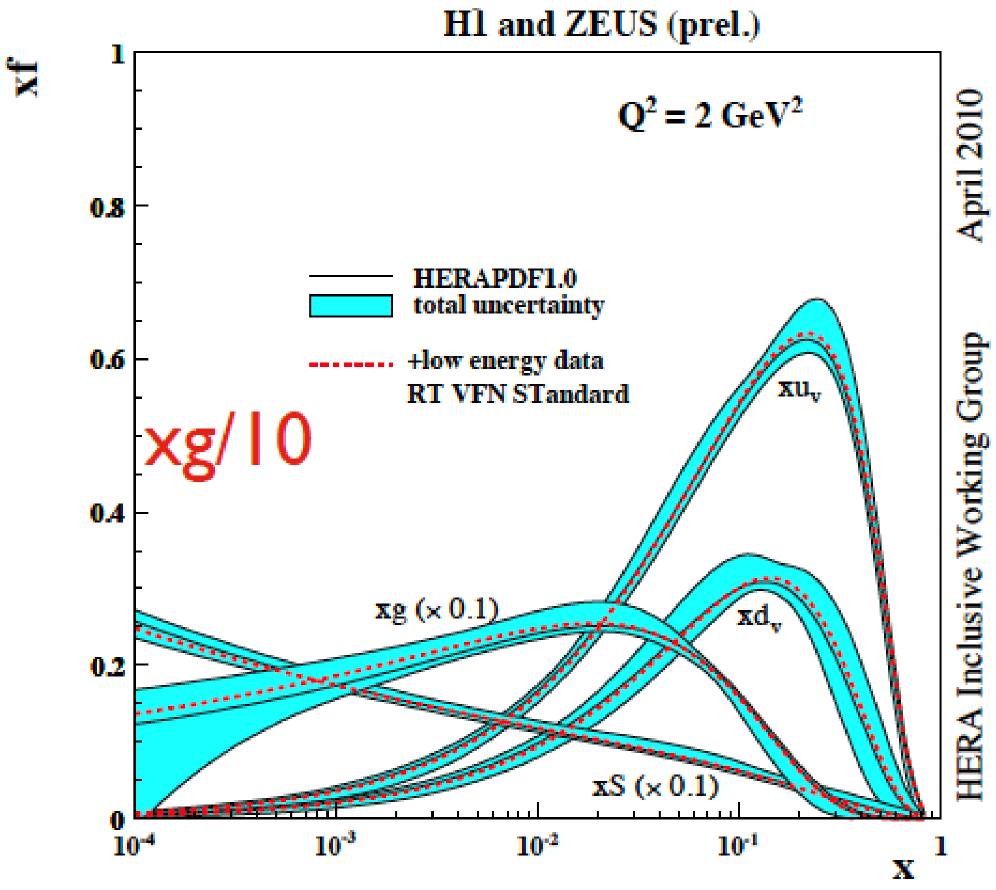
Supercomputers used to perform calculations

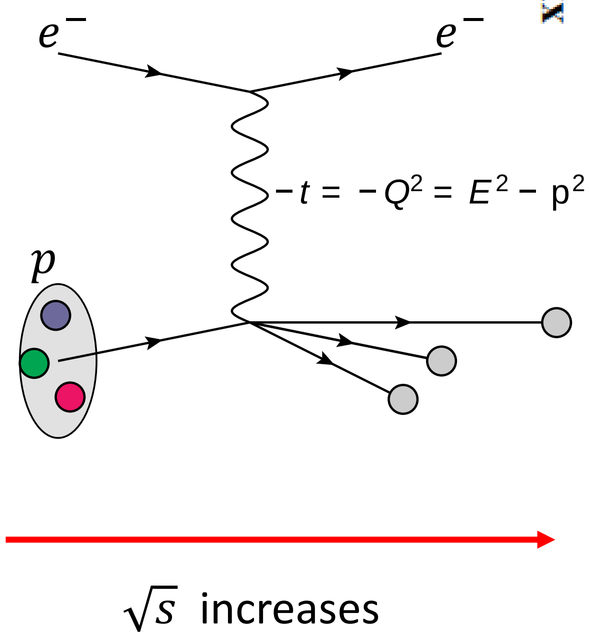


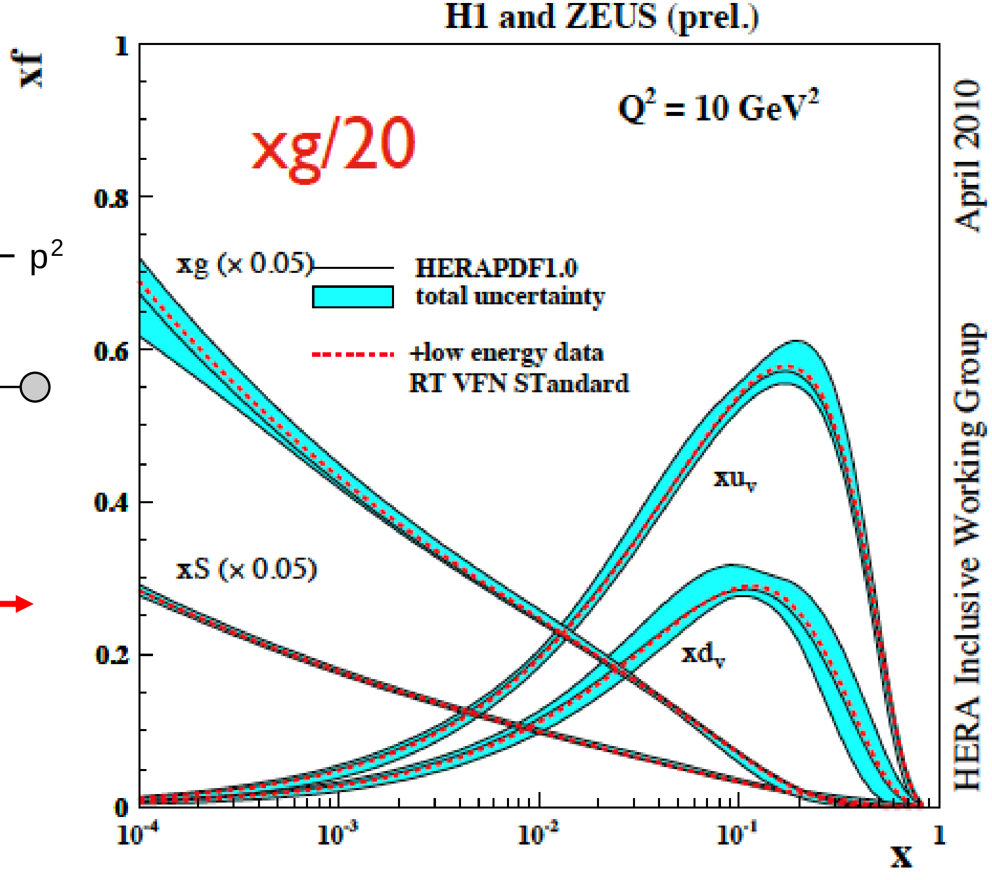
- Obtained an allocation of several million core-hours on Stampede 2 at Texas Advanced Computing Center
- Software setup, testing, and calculations are done over a 2-year period
- The simulations results used to train a *Gaussian Process Emulator* (GPE) that *efficiently* interpolates between calculated results
- The acceleration provided by the GPE is crucial to obtain the Bayesian Posterior.

Evolution of the particle composition at different $\sqrt{s}$

Parton distribution function (PDF) in a proton







- The relative contribution of gluons inside a proton ↑ as ↑ $\sqrt{s}$
- This relative excess of gluons persists once nuclear PDFs are used.

A modern approach to classical relativistic fluid dynamics

- As a solution to the classical relativistic Liouville equation on an n-particle phase space distribution doesn't yet exist ⇒ approximations are in order.

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- Instead of solving for each particle's distribution in the n-particle system, one solves for n-particle correlation functions (also useful for a QFT formulation).
    - This gives rise to ordered tower of coupled integro-differential equations in phase space known as the BBGKY (Bogoliubov–Born–Green–Kirkwood–Yvon) hierarchy, i.e. a systematic expansion scheme.

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  - Neglecting n-particle correlations $\forall\, n \geq 2$ ⇒ Boltzmann equation

$$p^\mu \partial_\mu f_p = C[f_p] \sim \int d\Phi^{2\to 2}_{p,k\to p'k'} \left|\mathcal{M}^{2\to 2}_{p,k\to p'k'}\right|^2 f_k f_{k'} f_{p'}$$

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  - Neglecting n-particle correlations $\forall\, n \geq 2$ ⇒ Boltzmann equation for a single-particle distribution

$$p^\mu \partial_\mu f_p = C[f_p] \sim \int d\Phi^{2\to 2}_{p,k\to p'k'} \left|\mathcal{M}^{2\to 2}_{p,k\to p'k'}\right|^2 f_k f_{k'} f_{p'}$$

- Expanding the Boltzmann equation up to second moment (rank-2 tensor) gives conservation equations of fluid dynamics

$$\partial_\mu T^{\mu\nu} = 0 \qquad\qquad T^{\mu\nu} = \langle p^\mu p^\nu \rangle = \int \frac{d^3p}{(2\pi)^3 p^0} p^\mu p^\nu f_p$$

where

$$\partial_\mu N^\mu = 0 \qquad\qquad N^\mu = \langle p^\mu \rangle = \int \frac{d^3p}{(2\pi)^3 p^0} p^\mu f_p$$

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  - This gives rise to ordered tower of coupled integro-differential equations in phase space known as the BBGKY (Bogoliubov–Born–Green–Kirkwood–Yvon) hierarchy, i.e. a systematic expansion scheme.
  - Neglecting n-particle correlations $\forall\, n \geq 2 \Rightarrow$ Boltzmann equation for a single-particle distribution

$$p^\mu \partial_\mu f_p = C[f_p] \sim \int d\Phi^{2\to 2}_{p,k\to p'k'} \left|\mathcal{M}^{2\to 2}_{p,k\to p'k'}\right|^2 f_k f_{k'} f_{p'}$$

- Expanding the Boltzmann equation up to second moment (rank-2 tensor) gives conservation equations of fluid dynamics

$$\partial_\mu T^{\mu\nu} = 0 \qquad\qquad T^{\mu\nu} = \langle p^\mu p^\nu \rangle = \int \frac{d^3p}{(2\pi)^3 p^0} p^\mu p^\nu f_p$$

where

$$\cancel{\partial_\mu N^\mu = 0 \qquad\qquad N^\mu = \langle p^\mu \rangle = \int \frac{d^3p}{(2\pi)^3 p^0} p^\mu f_p}$$

A modern approach to classical relativistic fluid dynamics

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- Boltzmann eq also EoMs for dissipative dofs…

An irreducible tensor decomposition of hydrodynamics

- In high-energy collisions (w/ negligible μ_B), what is flowing?... That can only be energy density ϵ

$$T^{\mu\nu}u_\nu = \epsilon u^\mu$$

Landau's flow definition

$u^\mu = \left(\gamma, \gamma\vec{\beta}\right)$ where

$\gamma = \left(1-\beta^2\right)^{-1/2}$ and $\vec{\beta} = \vec{v}/c$. Using natural units from how on $\Rightarrow c = 1$.

- Non-dissipative $T_0^{\mu\nu}$ can only take the form:

$$T_0^{\mu\nu} = \epsilon u^\mu u^\nu - P(\epsilon)\Delta^{\mu\nu} = \epsilon u^\mu u^\nu - P(\epsilon)(g^{\mu\nu} - u^\mu u^\nu)$$

- Including dissipation gives rise to dissipative corrections $\delta T^{\mu\nu}$ to $T_0^{\mu\nu}$, namely Π and $\pi^{\mu\nu}$

$$T^{\mu\nu} = T_0^{\mu\nu} + \delta T^{\mu\nu} = T_0^{\mu\nu} - \Pi\Delta^{\mu\nu} + \pi^{\mu\nu} = \epsilon u^\mu u^\nu - (P + \Pi)\Delta^{\mu\nu} + \pi^{\mu\nu}$$

- In the Navier-Stokes limit,

$$\pi_{NS}^{\mu\nu} = 2\eta\partial^{\langle\mu}u^{\nu\rangle} \qquad \Pi_{NS} = -\zeta\partial_\mu u^\mu$$

Already explained η...

ζ only ∃ in compressible fluids. It's the response of the fluid to abrupt radial compression.

- For incompressible fluids, rapid ↑ P_{ext} would ↑ translational motion of molecules (↑ T) and $\pi^{\mu\nu}$.
- For compressible fluids, rapid ↑ P_{ext} can also excite rotational and vibrational motion of molecules, which is incorporated in Π.

Relativistic dissipative hydrodynamics from Grad's expansion

- Relativistic dissipative hydrodynamics

$$\partial_\mu T^{\mu\nu} = 0$$

$$T^{\mu\nu} = \varepsilon u^\mu u^\nu - [P(\varepsilon) + \Pi]\Delta^{\mu\nu} + \pi^{\mu\nu}$$

$$\Delta^{\mu\nu} = g^{\mu\nu} - u^\mu u^\nu,\ \sigma^{\mu\nu} = \partial^{\langle\mu}u^{\nu\rangle},\ \theta = \partial_\mu u^\mu$$

Boltzmann equation gives

$$\tau_\Pi \dot{\Pi} + \Pi = -\zeta\theta - \delta_{\Pi\Pi}\Pi\theta + \lambda_{\Pi\pi}\pi^{\mu\nu}\sigma_{\mu\nu}$$

$$\tau_\pi \dot{\pi}^{\langle\mu\nu\rangle} + \pi^{\mu\nu} = 2\eta\sigma^{\mu\nu} - \delta_{\pi\pi}\pi^{\mu\nu}\theta + \phi_7\pi_\alpha^{\langle\mu}\pi^{\nu\rangle\alpha} - \tau_{\pi\pi}\pi_\alpha^{\langle\mu}\sigma_\alpha^{\nu\rangle} + \lambda_{\pi\Pi}\Pi\sigma^{\mu\nu}$$

- $P(\varepsilon)$ use lattice EoS, and the goal is to constrain ζ and η via Bayesian analysis [all transport coefficient are set c.f. PRD **85** 114047 (2012), PRC **90** 024912 (2014)]
- About power counting: the r.h.s. of the PDE for Π and $\pi^{\mu\nu}$ contain up to 2nd order terms, in powers of two small quantities: [J. Phys. G: Nucl. Part. Phys. 41, 124004 (2014)]
  - <u>Knudsen number</u>: $K_n = \frac{\lambda_{mfp}}{L}$ powers in microscopic scale (λ_{mfp}) and macroscopic scale (L). $2\eta\sigma^{\mu\nu}$: $\eta \sim \lambda_{mfp}$ while $\sigma^{\mu\nu} = \partial^{\langle\mu}u^{\nu\rangle} \sim \frac{1}{L} \Rightarrow K_\pi = 2\eta\sigma^{\mu\nu} \ll 1$ is first order K_π and so is $K_\Pi = -\zeta\theta \ll 1$.

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  - inverse Reynolds number: counts powers of dissipative forces over equilibrium forces: $\Pi \ll P$, so Π is first order in $R_\Pi^{-1} = \frac{|\Pi|}{P} \ll 1$ and so is $R_\pi^{-1} = \frac{|\pi^{\mu\nu}|}{P} \ll 1$.

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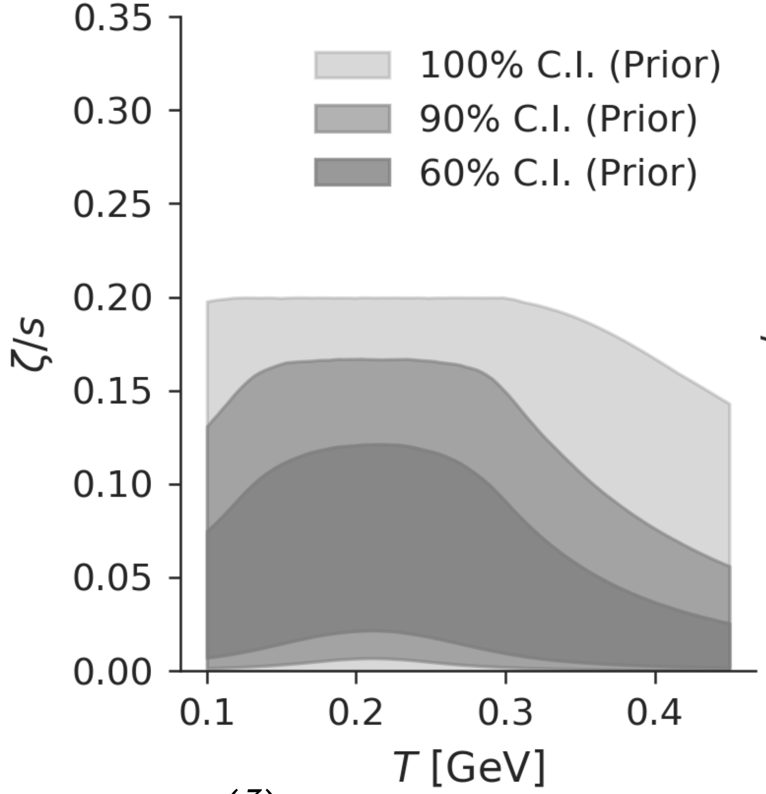
  - In EoM for $\Pi, \pi^{\mu\nu}$ above, two kinds of second order terms ∃: $\delta_{\Pi\Pi}\Pi\theta \sim K_\Pi R_\Pi^{-1}$ while $\phi_7 \pi_\alpha^{\langle\mu}\pi^{\nu\rangle\alpha} \sim R_\pi^{-2}$

Bayesian Prior for bulk (ζ/s) and shear (η/s) viscosities

- Bulk and shear viscosities were parametrized using 4-parameter functions

PRC **103** 054904 (2021)



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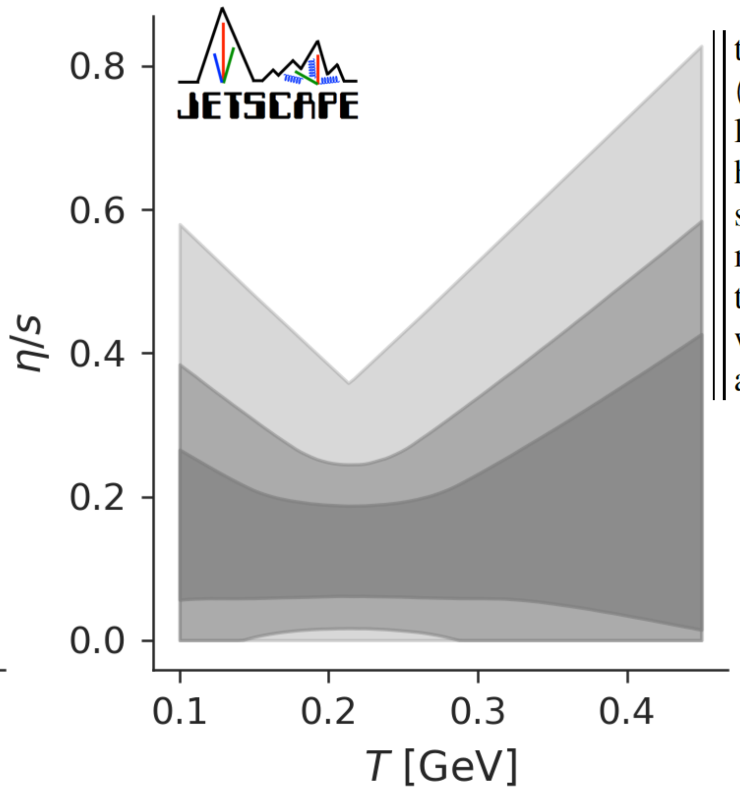

PRC **103** 054904 (2021)

temperature of (η/s) kink	T_η	[0.13, 0.3] GeV
(η/s) at kink	$(\eta/s)_{\text{kink}}$	[0.01, 0.2]
low temp. slope of (η/s)	a_{low}	[–2, 1] GeV^{-1}
high temp. slope of (η/s)	a_{high}	[–1, 2] GeV^{-1}
shear relaxation time factor	b_π	[2, 8]
maximum of (ζ/s)	$(\zeta/s)_{\max}$	[0.01, 0.25]
temperature of (ζ/s) peak	T_ζ	[0.12, 0.3] GeV
width of (ζ/s) peak	w_ζ	[0.025, 0.15] GeV
asymmetry of (ζ/s) peak	λ_ζ	[–0.8, 0.8]

$$\frac{\zeta}{s}(T) = \frac{\left(\frac{\zeta}{s}\right)_{\max}\Lambda^2}{\Lambda^2 + \left(T - T_\zeta\right)^2}$$

$$\Lambda = w_\zeta\left[1 + \lambda_\zeta\left(T - T_\zeta\right)\right]$$

$$\frac{\eta}{s}(T) = a_{\text{low}}\left(T - T_\eta\right)\Theta\left(T_\eta - T\right) + \left(\frac{\eta}{s}\right)_{\text{kink}} + a_{\text{high}}\left(T - T_\eta\right)\Theta\left(\mathrm{T} - \mathrm{T}_\eta\right)$$