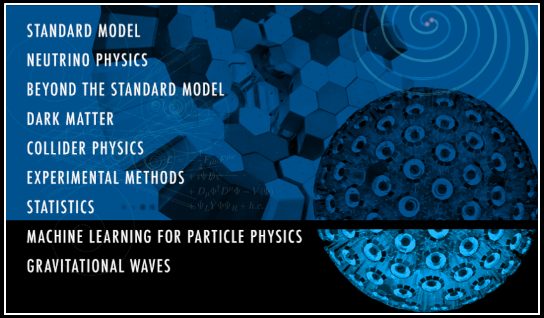


TRISEP lectures on Gravitational Waves

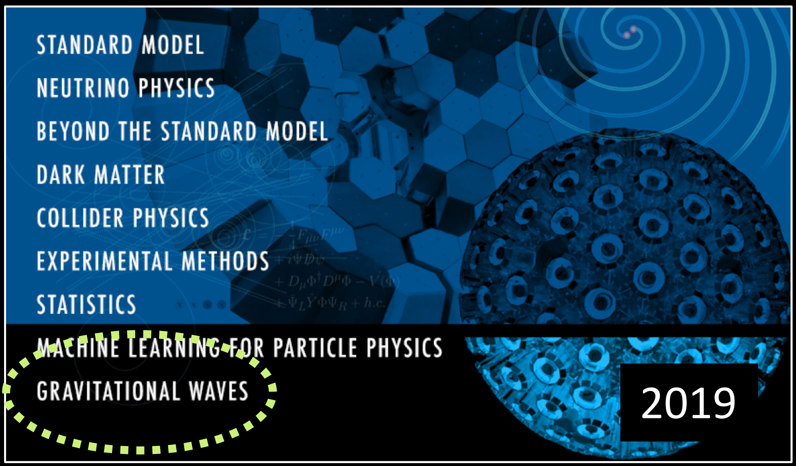
Djuna Croon, TRIUMF

A new era of astronomy!

- First measurement of a binary black hole merger in 2015 by LIGO
- LIGO/Virgo have detected 11+19 mergers
- Many new experiments planned

- Huge opportunity for (particle) astrophysics and cosmological research!

A new era of particle theory!



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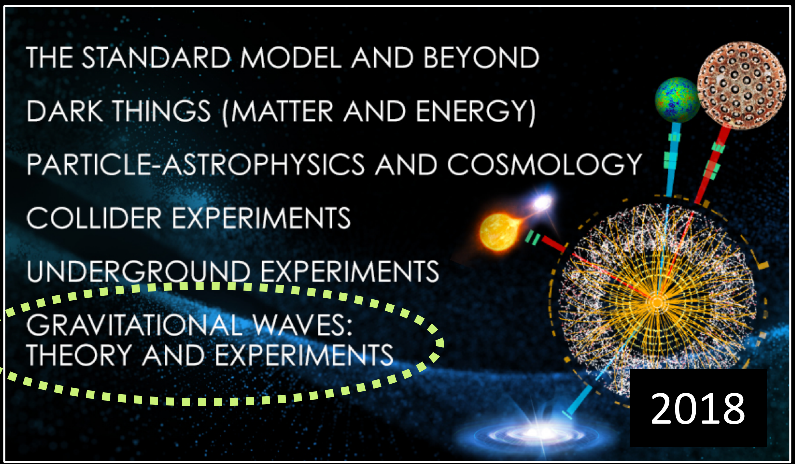
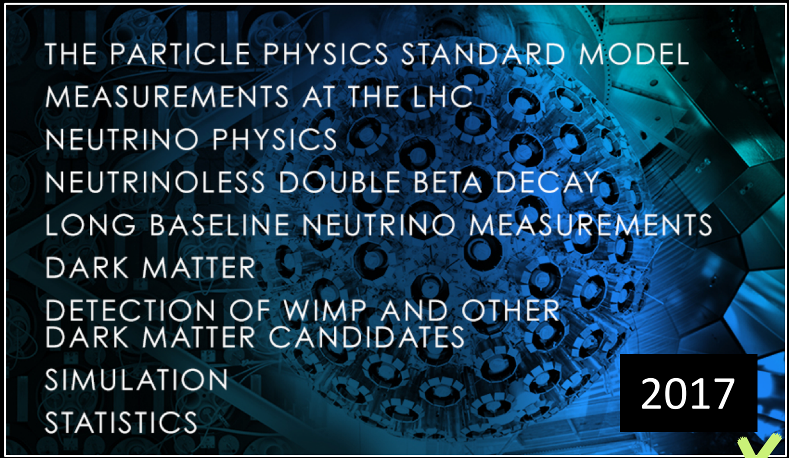
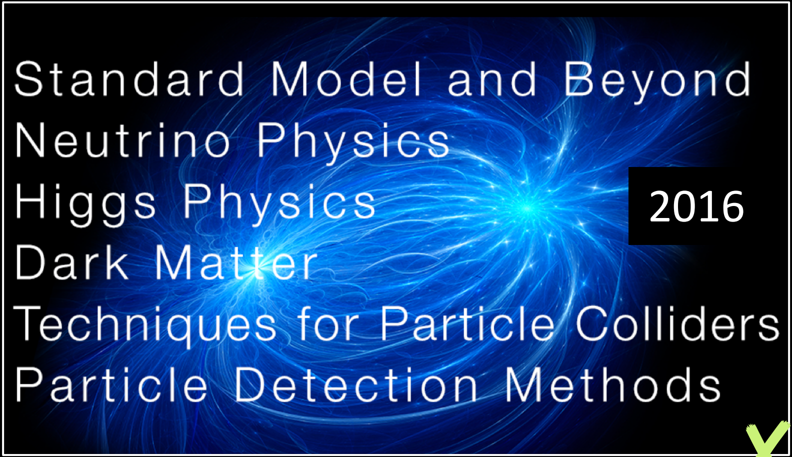






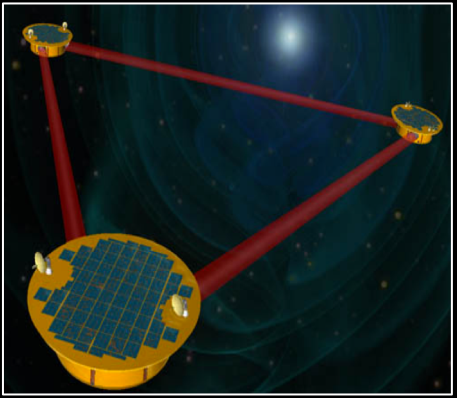
Gravitational wave timeline

Poincaré, Sur la dynamique de l'electron, 1905

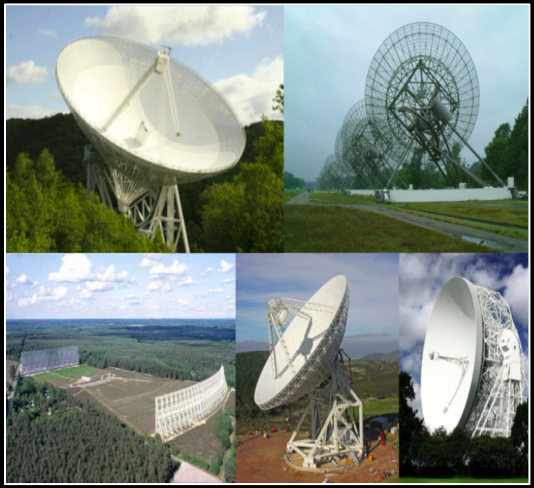
- Proposed in 1905 by Poincaré
- Predicted in 1916 by Einstein
- Indirect evidence in 1974 from the Hulse-Taylor binary pulsar (1993 Nobel Prize)
- Direct evidence (2015 onwards)
  - Interferometer proposals: 1960s
  - First detection in 2015 (announced in 2016) by the LIGO collaboration (2017 Nobel Prize)

Current/future experiments

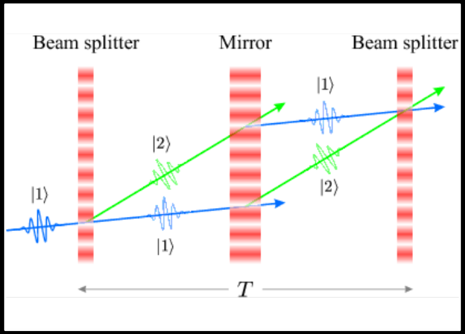
Ground-based interferometers
LIGO, Virgo, KAGRA (now)
ET, CE (ca. 2030)



Space-based interferometers
LISA, Tianqin, Decigo, BBO
(ca. 2035+)



Pulsar timing Arrays
EPTA, IPTA, SKA
(now)



Atom interferometry
AION, MAGIS
(ca. 2025)

These lectures

- A (brief) note on General Relativity
- Gravitational wave theory
- Binary Mergers
- Detection
- Science opportunities and prospects

A (brief) note on

GENERAL RELATIVITY

Scalars, vectors, and tensors

- I will assume you are familiar with index notation:

A	Scalar
A_μ	Vector
$A_{\mu\nu}$	Tensor (rank 2)

- Our indices will (generally) run over space and time variables: $\mu, \nu = \{t, x, y, z\}$

Einstein notation

- I will also use the following notation:
  - Covariant vector: A_μ
  - Contravariant vector: A^μ
- Greek indices (μ, ν) run over spacetime, Latin indices (i, j) run over space
- Repeated indices are summed over,

$$x^i x_i \equiv \sum_i x^i x_i$$

“Spacetime tells matter how to move;
matter tells spacetime how to curve”

John A. Wheeler

The Einstein Field Equations

$$G_{\mu\nu} \equiv R_{\mu\nu} - \tfrac{1}{2} R \, g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

The Einstein Field Equations

Einstein tensor

Ricci scalar

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R \, g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

Ricci tensor

Metric

Energy-momentum tensor

The Einstein Field Equations

Einstein tensor

Ricci scalar

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R \, g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

Ricci tensor

Energy-momentum tensor

Metric

Curvature

Matter

The metric tensor

- Symmetric real rank-2 tensor $g_{\mu\nu} = g_{\nu\mu}$

$$g_{\mu\nu} = \begin{pmatrix} g_{tt} & g_{tx} & g_{ty} & g_{tz} \\ g_{xt} & g_{xx} & g_{xy} & g_{xz} \\ g_{yt} & g_{yx} & g_{yy} & g_{yz} \\ g_{zt} & g_{zx} & g_{zy} & g_{zz} \end{pmatrix}$$

- Measures distance: $ds^2 = \sum_{\mu,\nu} g_{\mu\nu} dx^\mu dx^\nu$

Q: How many independent components does $g_{\mu\nu}$ have (maximally)?

The metric tensor

- Symmetric real rank-2 tensor $g_{\mu\nu} = g_{\nu\mu}$

$$g_{\mu\nu} = \begin{pmatrix} g_{tt} & g_{tx} & g_{ty} & g_{tz} \\ g_{xt} & g_{xx} & g_{xy} & g_{xz} \\ g_{yt} & g_{yx} & g_{yy} & g_{yz} \\ g_{zt} & g_{zx} & g_{zy} & g_{zz} \end{pmatrix}$$

- Measures distance: $ds^2 = \sum_{\mu,\nu} g_{\mu\nu} dx^\mu dx^\nu$

A: 10

Example: 2 spatial dimensions

$\mu, \nu = \{x, y\}$

- Distance measured as:

$$ds^2 = \sum_{\mu,\nu} g_{\mu\nu} dx^\mu dx^\nu$$

$$= g_{xx} dx^2 + g_{yy} dy^2 + 2g_{xy} dx dy$$

Pythagoras theorem

- Flat space: $ds^2 = dx^2 + dy^2$
  So, in flat *space*, $g_{ii} = 1, \ g_{ij} = 0 \text{ for } i \neq j$

- Flat space-*time* has $g_{\mu\nu} = \eta_{\mu\nu} = \pm diag(\text{-}1,1,1,1)$

The Ricci tensor $R_{\mu\nu}$ and scalar R

- Describe the geometry of space-time
- Derived from the Riemann tensor, $R^{\mu}{}_{\nu\sigma\rho}$

Ricci tensor: contract the first and the third index
Ricci scalar: contract the Ricci tensor (with $g^{\mu\nu}$)

- Flat spacetime: $R_{\mu\nu} = 0 = R$
  - But, remember the EFE:

$$R_{\mu\nu} - \frac{1}{2} R \, g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

  - Flat spacetime is empty!

Matter ←→ curvature

- In reality, space-time is *almost* flat almost everywhere. Gravity is weak,

$$\frac{8\pi G_N}{c^4} = 2.1 \times 10^{-43}\,\mathrm{s}^2\,\mathrm{kg}^{-1}\,\mathrm{m}^{-1}$$

- For example, consider the sun:

$$\text{mass density of the sun} = 1.4\,\mathrm{g}\,\mathrm{cm}^{-3} \times c^2$$

$$\text{energy density of the sun} = 1.3 \times 10^{20}\,\mathrm{kg}\,\mathrm{m}^{-1}\,\mathrm{s}^{-2}$$

$$G_{\mu\nu} = 2.6 \times 10^{-23}\,\mathrm{m}^{-2}$$

~ (radius of curvature)$^{-2}$

Matter ←→ curvature

- In reality, space-time is *almost* flat almost ev

Learn more: Repeat this exercise for different astrophysical systems (for example the Earth)

- For example, consider the s

$$\text{mass density of the sun} = 1.$$

$$\text{energy density of the sun} = 1.3 \times 10^{20}\,\text{kg}\,\text{m}^{-1}\,\text{s}^{-2}$$

$$G_{\mu\nu} = 2.6 \times 10^{-23}\,\text{m}^{-2}$$

~ (radius of curvature)$^{-2}$

An introduction to

GRAVITATIONAL WAVES

What is a ~~gravitational~~ wave?

- A solution to a wave equation:

$$\Box\, h(\overrightarrow{x}, t) = \left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial \overrightarrow{x}^2}\right] h(\overrightarrow{x}, t) = 0$$

- Or, with a source:

$$\Box\, h(\overrightarrow{x}, t) = [\text{source}]$$

- We will see that the EFE take this form in linearized theory

Linearized GR

- As we saw, the (Minkowski) metric of flat space-time is given by $\eta_{\mu\nu} = \pm diag(1,-1,-1,-1)$

- Imagine that

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \quad \text{where} \quad h_{\mu\nu} \ll g_{\mu\nu}$$

  i.e., a flat metric with a small perturbation

- Now we fill this into the EFE and perform some dark magic (Convenient gauge changes)

EFE for a metric perturbation*

$$\partial^\nu h_{\mu\nu} = 0$$

Lorentz gauge

$$\Box h_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}$$

Metric perturbation

Source: energy momentum tensor

Q: How many independent components does $h_{\mu\nu}$ have?

* This is actually an equation for the trace-reversed metric perturbation, but for our purposes the difference is not important

EFE for a metric perturbation*

$$\partial^\nu h_{\mu\nu} = 0$$

Lorentz gauge

$$\Box h_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}$$

Metric perturbation

Source: energy momentum tensor

A: 6 = 10 (symmetric tensor)
- 4 (Lorentz gauge)

* This is actually an equation for the trace-reversed metric perturbation, but for our purposes the difference is not important

Transverse-Traceless gauge

- Outside of the source, $\Box h_{\mu\nu} = 0$
  - This gives 4 more conditions
  - $h_{\mu\nu}$ has 6-4=2 independent components
- This is exploited in the Transverse-Traceless gauge,

$$
\begin{aligned}
h_{\mu 0} &= 0 && \text{only spatial components} \\
h_j{}^j &= 0 && \text{traceless} \\
h_{ij,}{}^j &= 0 && \text{no divergence}
\end{aligned}
$$

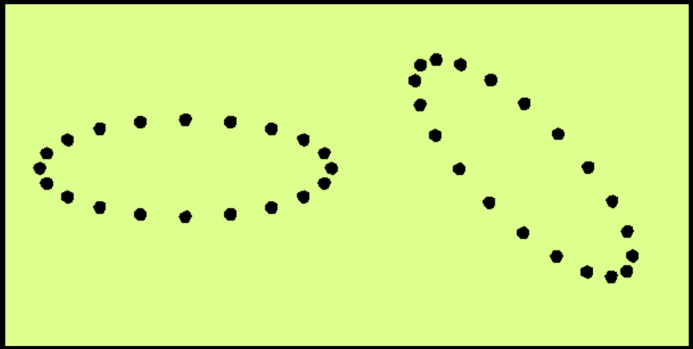
Learn more: find an example of a metric in the TT gauge

GW polarization

- Example: wave traveling down the z-axis

$$h_{ab}^{TT}(t,z) = \left(\begin{array}{cc} h_+ & h_\times \\ h_\times & -h_+ \end{array} \right) \cos\left(\omega(t - z/c)\right)$$

- Z-axis into/out of the slide:



Solving $\Box h_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}$

- Recall that generally, linear wave equations can be solved using Green's functions:

$$\Box_x G(x - x') = \delta^4(x - x')$$

- Just as in electrodynamics, we need the *retarded* Green's function (traveling forward in time)
- The solution is then,

Learn more: Verify this

$$h_{ij}^{TT}(t, \vec{x}) = \frac{4G}{c^4} \Lambda_{ij,kl} \int d^3x' \, \frac{1}{|\vec{x} - \vec{x}'|} T_{kl}\left(t - \frac{|\vec{x} - \vec{x}'|}{c}, \vec{x}'\right)$$

TT-projector

$$\Lambda_{ij,kl} = P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl}$$

Q: why is the integral over space only?

Solving $\Box h_{\mu\nu} = -\frac{16\pi G}{c^4} T_{\mu\nu}$

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↑ TT-projector

$$\Lambda_{ij,kl} = P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl}$$

A: the time elements are related by energy-momentum conservation

Further approximations

- To study $h_{\mu\nu}$ further, we will take two limits:
  1. The detector is far from the source
  2. The source is non-relativistic
- The detector is far (1): we can expand

$$|\overrightarrow{x} - \overrightarrow{x}'| = |\overrightarrow{r}| - \overrightarrow{x}' \cdot \hat{N}$$

Q: for a source of size d, why is $x' \leq d$?

$$h_{ij}^{TT}(t, \overrightarrow{x}) = \frac{1}{r}\frac{4G}{c^4}\Lambda_{ij,kl}\int d^3x'\, T_{kl}\left(t - \frac{|\overrightarrow{r}|}{c} + \frac{\overrightarrow{x}' \cdot \hat{N}}{c}, \overrightarrow{x}'\right)$$

Further approximations

- To study $h_{\mu\nu}$ further, we will take two limits:
  1. The detector is far from the source
  2. The source is non-relativistic
- The detector is far (1): we can expand

$$|\vec{x} - \vec{x}'| = |\vec{r}| - \vec{x}' \cdot \hat{N}$$

A: $T_{\mu\nu}$ is only nonzero inside d

$$h_{ij}^{TT}(t, \vec{x}) = \frac{1}{r}\frac{4G}{c^4}\Lambda_{ij,kl}\int d^3x'\, T_{kl}\left(t - \frac{|\vec{r}|}{c} + \frac{\vec{x}' \cdot \hat{N}}{c}, \vec{x}'\right)$$

Weak field, low velocity

- For self-gravitating systems,

$$\mu = \frac{\prod_i m_i}{m_{\rm tot}}$$

Reduced mass

$$\frac{1}{2}\mu v^2 = \frac{1}{2}\frac{G_N \mu m_{\rm tot}}{r}$$

- Such that the weak-field limit $(R_s \ll r)$ implies the low-velocity limit,

$$R_s = \frac{2 G_N m_{\rm tot}}{c^2}$$

Schwarzschild radius

$$\frac{v^2}{c^2} = \frac{1}{2}\frac{R_s}{r}$$

Low velocity expansion (2)

- Imagine a source of size d and frequency $\omega,$ such that the linear velocity is $v = \omega d$
- As we will see, the GW frequency is then also $\omega_{GW} = O(\omega)$, such that

$$\lambda_{\mathrm{GW}} \sim \frac{c}{v} d$$

- For NR systems $(c \gg v)$, we find $\lambda_{\mathrm{GW}} \gg d$
- Internal motions unimportant ➔ multipole expansion converges

Multipole expansion

- Using the expansion,

$$T_{kl}\left(t-\frac{|\overrightarrow{r}|}{c}+\frac{\overrightarrow{x}'\cdot\hat{N}}{c},\overrightarrow{x}'\right)=T_{kl}+\frac{x'^{i}n_i}{c}\partial_0 T_{kl}+\ldots\Big|_{(t-\frac{r}{c},\overrightarrow{x}')}$$

- We can express h_{ij} in moments of T_{ij}

$$h_{ij}^{TT}(t,\overrightarrow{x})=\frac{4G}{c^4}\Lambda_{ij,kl}\times\left(S^{kl}+\frac{n_m}{c}\dot{S}^{kl,m}+\ldots\right)$$

$$S^{ij}(t)=\int d^3x\, T^{ij}(t,x)$$

$$S^{ij,k}(t)=\int d^3x\, T^{ij}(t,x)x^k$$

First two moments of T_{ij}

Mass quadrupole moment

- It can be shown using $T_{\mu\nu,}{}^{\nu} = 0$

$$S^{ij} = \int d^3x\, T^{ij}(t,x) = \partial_0^2 \left[\frac{1}{c^2} \int d^3x\, T^{00}(t,x) x^i x^j \right] = \ddot{M}_{ij}$$

Learn more: Show this. Hint: remember that $T_{\mu\nu}$ vanishes outside the source

- Here M_{ij} is the mass quadrupole moment

$$h_{ij}^{TT} = \left[h_{ij}^{TT}\right]_{\text{quad}} + \ldots$$

$$\left[h_{ij}^{TT}\right]_{\text{quad}} = \frac{1}{r}\frac{2G}{c^4}\Lambda_{ij,kl}\ddot{M}_{kl}(t - r/c)$$

Take-home message

- Gravitational waves are generated by accelerated mass distributions with a nonzero mass quadrupole moment
  - No spherically symmetric systems
  - No static or uniformly moving systems

- Observable GW sources are huge and relatively close by (or very numerous)

Take-home message

- We found the first term in the expansion to be the quadrupole moment,

$$\left[h_{ij}^{TT}\right]_{\text{quad}} = \frac{1}{r}\frac{2G}{c^4}\Lambda_{ij,kl}\ddot{M}_{kl}(t - r/c)$$

- Let's plug in some numbers…

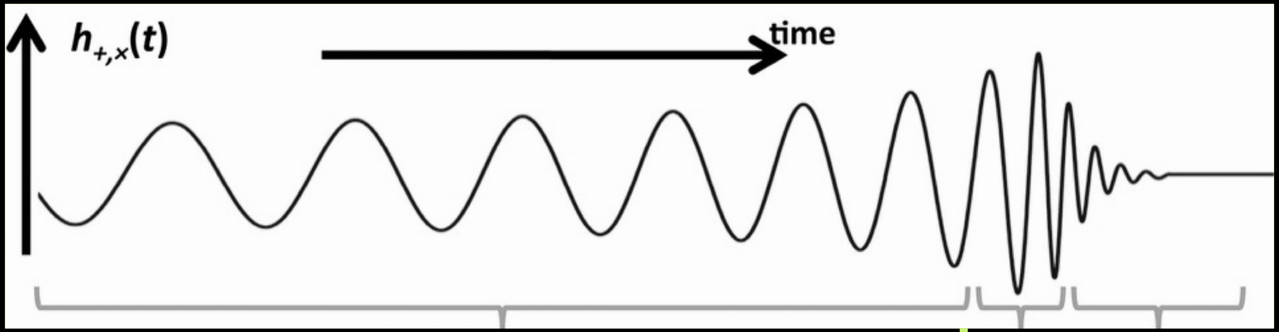
$$\left.\begin{aligned} r &= 140 \times 10^6\ \text{ly} \\ \ddot{M}_{kl} &= 60 M_\odot \times c^2 \end{aligned}\right\} \left[h_{ab}^{TT}\right]_{\text{quad}} \sim 10^{-19}$$

Gravitational waves from

BINARY MERGERS

Binary mergers

The GW observed at LIGO/Virgo are from the inspiral phases of BNS and BBH mergers



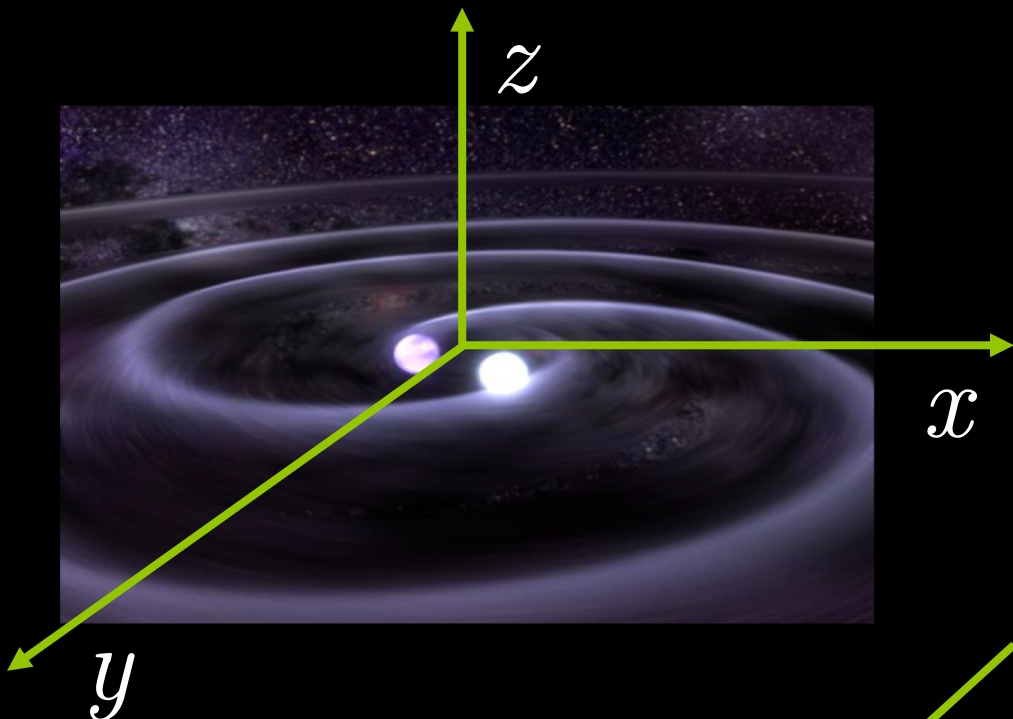
Inspiral phase

Merger

Ringdown

In the inspiral phase, the approximations from the previous section are good

$$f_{\rm ISCO} = \frac{C_*^{3/2}}{3^{3/2}\,\pi\, G_N (M_1 + M_2)}$$



We choose the origin of our coordinate system to be a the center of mass (CM) of the binary

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

Reduced mass

Mass density CM frame

$$\rho(t, \overrightarrow{x}) = \mu\, \delta^{(3)}\left(\overrightarrow{x} - \overrightarrow{x}_0(t)\right)$$

Learn more: Show this.

$$M^{ij}(t) = \mu x_0^i(t) x_0^j(t)$$

Polarization waveforms

Let's first imagine the wave propagation along the z-axis:

$$\left[h_{ij}^{TT}\right]_{\rm quad} = \frac{1}{r}\frac{2G}{c^4}\Lambda_{ij,kl}\ddot{M}_{kl}(t-r/c)$$

Using the TT-projector,

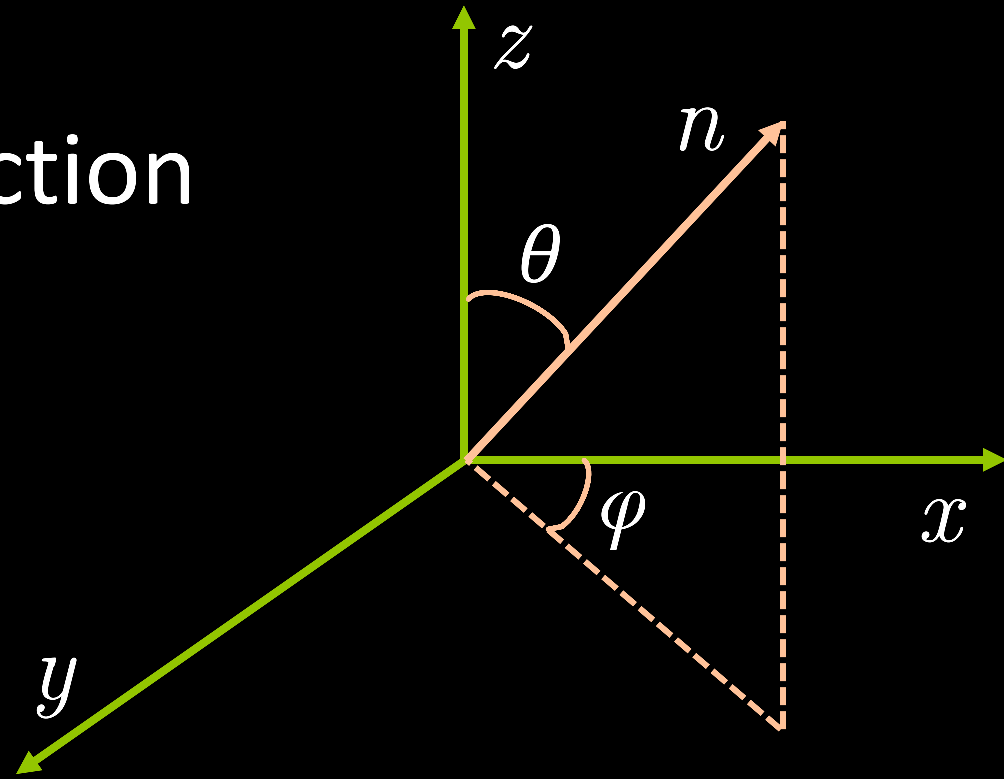
$$\Lambda_{ij,kl} = P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl}$$

$$h_+ = \frac{1}{r}\frac{G}{c^4}\left(\ddot{M}_{11} - \ddot{M}_{22}\right)$$

$$h_\times = \frac{2}{r}\frac{G}{c^4}\ddot{M}_{12}$$

We can rotate our result to find results for other propagation directions

More general propagation direction



Use the rotation matrix to translate the previous result to a general direction

$$h_+ = \frac{1}{r}\frac{G}{c^4}\left(\ddot{M}_{11}\left(\cos^2\phi - \sin^2\phi\cos^2\theta\right) + \ddot{M}_{22}\left(\sin^2\phi - \cos^2\phi\cos^2\theta\right) - \ddot{M}_{12}\sin 2\phi\left(1+\cos^2\theta\right)\right)$$

We chose an orbit in the (x,y)-plane, hence $M_{i3}=0$

$$h_\times = \frac{2}{r}\frac{G}{c^4}\left(\left(\ddot{M}_{11} - \ddot{M}_{22}\right)\sin 2\phi\cos\theta + \ddot{M}_{12}\cos 2\phi\cos\theta\right)$$

A simplified calculation

Further assumptions:

- $m_1 = m_2$
- Circular orbits
- No backreaction

Source motion:

$$x_0(t) = -R\sin(\omega t)$$

$$y_0(t) = R\cos(\omega t)$$

$$M^{ij}(t) = \mu x_0^i(t) x_0^j(t)$$

Learn more: Show this.

$$h_+ = \frac{1}{r}\frac{4G\mu\omega^2 R^2}{c^4}\left(\frac{1+\cos^2\theta}{2}\right)\cos(2\omega t + 2\phi)$$

$$h_\times = \frac{2}{r}\frac{4G\mu\omega^2 R^2}{c^4}\cos\theta\sin(2\omega t + 2\phi)$$

A simplified calculation

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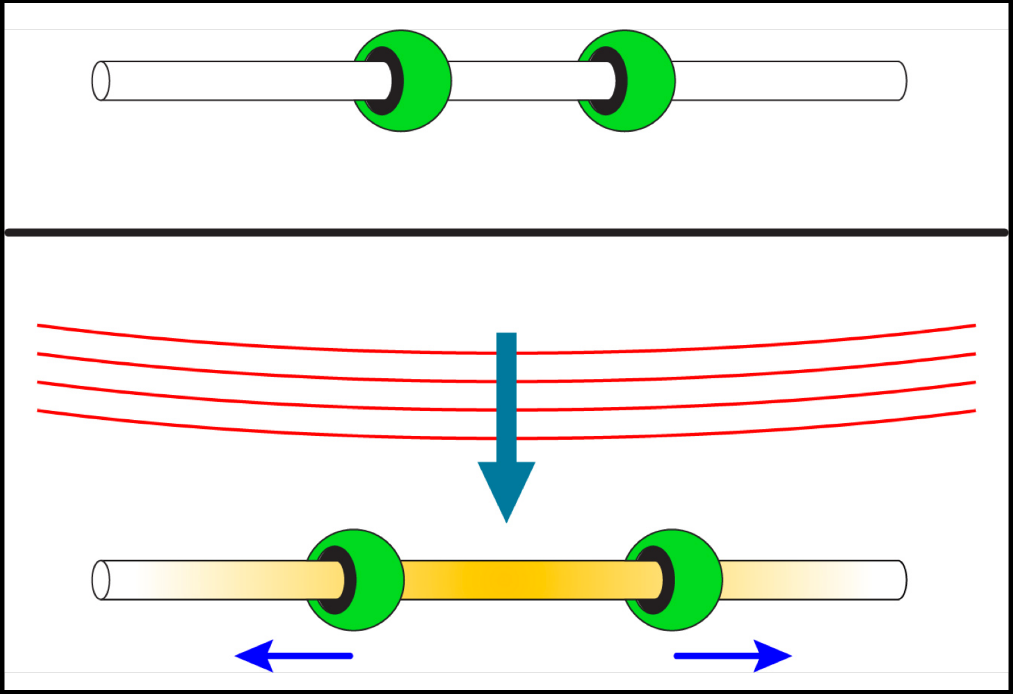
$$h_+ = \frac{1}{r}\frac{4G\mu\omega^2 R^2}{c^4}\left(\frac{1+\cos^2\theta}{2}\right)\cos(2\omega t + 2\phi)$$

$$h_\times = \frac{2}{r}\frac{4G\mu\omega^2 R^2}{c^4}\cos\theta\sin(2\omega t + 2\phi)$$

Twice the source frequency!

In reality, there *is* backreaction

- Gravitational waves carry energy away from the (binary) system
- Settled in 1957 with the sticky bead argument



In reality, there *is* backreaction

- Orbital frequency: Kepler's 3rd law

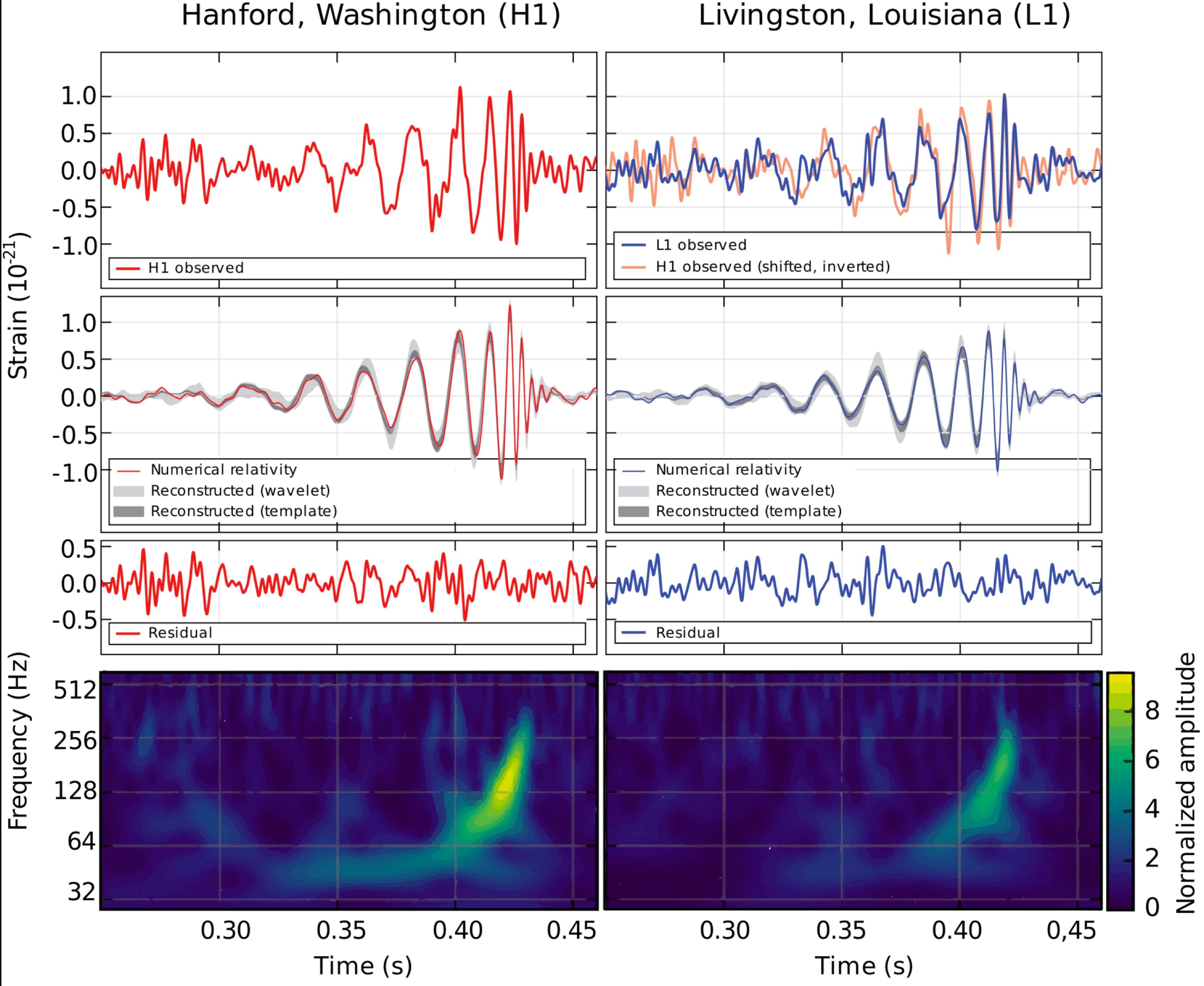
$$\omega^2 = G_N \frac{m_1 + m_2}{r^3}$$

- GW emission drains energy from the system,

$$P_{\rm GW} = \dot{E}_{\rm orbit}$$

$$E_{\rm orbit} = E_{\rm kin} + E_{\rm pot} = -G\,\frac{m_1 m_2}{2r}$$

GW emission implies that the orbital radius decreases and the frequency increases


Hanford, Washington (H1)
Livingston, Louisiana (L1)
Strain (10^{-21})
1.0
0.5
0.0
-0.5
-1.0
H1 observed
L1 observed
H1 observed (shifted, inverted)
Numerical relativity
Reconstructed (wavelet)
Reconstructed (template)
Residual
Frequency (Hz)
512
256
128
64
32
0.30
0.35
0.40
0.45
0,45
Time (s)
Normalized amplitude
8
6
4
2
0