Relativistic Generalized Uncertainty Principle and Minimum Length

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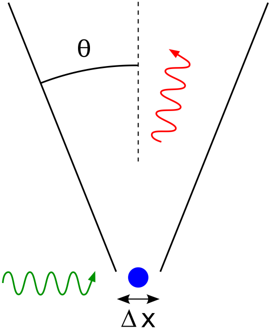
Outline

1. Generalized Uncertainty Principle from Quantum Gravity
2. Problems : Frame-dependence, Composition laws
3. Relativistic Generalized Uncertainty Principle [1]
4. Applications
5. Summary and outlook

1. V. Todorinov, P. Bosso, S. Das, Ann. Phys. **405**, 92-100 (2019) [arXiv :1810.11761].

Minimum length from Quantum Gravity

- Considering the Heisenberg thought experiment



from classical photon scattering	$\Delta x \geq \lambda / \sin\theta$
from Compton recoil	$\Delta p \geq h \sin\theta / \lambda$ h is Planck constant
then the uncertainty relation we get is	$\Delta x \Delta p \geq h$

Image Credit : Heisenberg_gamma_ray_microscope.png by WillowW

Heisenberg microscope with Newtonian potential

- This leads to uncertainty in its position

$$\Delta x \approx \frac{Gh}{\lambda c^3} \sin\theta$$

- Into the potential of the particle the photon changes momentum

$$\Delta p_\gamma = mGh/r\lambda c^2$$

- This corresponds to the uncertainty in the momentum of the particle

$$\Delta p \geq \frac{h}{\lambda}\left(1 + \frac{Gm}{rc^2}\right) \sin\theta$$

- Summing all the uncertainties for the position we arrive at

$$\Delta x \Delta p \geq \left(h + \underbrace{\frac{Gh}{c^3}}_{=l_{Pl}^2} \frac{\Delta p^2}{h} \right)$$

As one can see that this is a modification of the Heisenberg uncertainty principle , known as Generalized Uncertainty Principle

Generalized Uncertainty Principle (GUP)

- *"Hilbert space representation of the minimal length uncertainty relation" published in 1995*[2]

$$[x, p] = i\hbar(1 + \beta p^2)$$

- This amounts to the following minimum uncertainty in position

$$\Delta x_{\text{min}} = \hbar\sqrt{\beta}$$

- In *n*-dimensions

$$[x_i, p_j] = i\hbar\delta_{ij}(1 + \beta\vec{p}^2)$$

- Non-commutative position operators

$$[x_i, x_j] = 2i\hbar\beta(p_i x_j - p_j x_i)$$

2. A. Kempf, G. Mangano, and R. B. Mann, Phys. Rev.**D52**, 1108 (1995) [arXiv :hep-th/9412167]

GUP is robust. References :

- F. Scardigli, Phys. Lett. B, **452** :39–44, 1999.
- L. J. Garay, Int. J. Mod. Phys. A, **10(02)** :145–165, 1995.
- D. Amati, M. Ciafaloni, and G. Veneziano, Phys. Lett. B, **216(1)** :41–47, 1989.
- C. Rovelli and L. Smolin, Nucl. Phys., Sect. B, **442(3)** :593–619, may 1995.
- C. A. Mead, Phys. Rev., **135(3B)** :B849–B862, 1964.
- M. Maggiore, Phys. Lett. B, 304(1) :65–69, 1993.
- E. Witten, Phys. Tod. **49**, 4, 24 (1996).

Frame-Dependence

- Length is **not** a Lorentz invariant
- Generalized Uncertainty Principle adopt this form

$$[x_i, p_j] = i\hbar\delta_{ij}(1 + \beta\vec{p}^2) + \beta_1 p_i p_j$$

where $i, j \in \{1, 2, 3\}$,

- We need Lorentz covariant minimum length for constructing Quantum Field Theories with minimum length

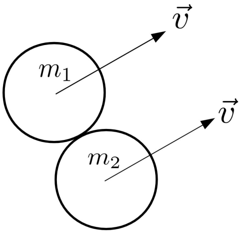
Composition law problem

- Modified dispersion relation

$$E^2 = p^2c^2 + m^2c^4 \;\rightarrow\; E^2 = p^2c^2 + m^2c^4 + f(p^2)$$

- It can be shown that for a composite system, moving at the same speed

$$\vec{P}_3 \neq \vec{P}_2 + \vec{P}_1 \qquad E_3 \neq E_1 + E_3$$



Relativistic Generalized Uncertainty Principle (RGUP)

- Most general quadratic relativistic GUP $\gamma = 1/(M_{Pl}c)$

$$[x^\mu, p^\nu] = i\hbar \left(1 + (\varepsilon - \alpha)\gamma^2 p^\rho p_\rho\right) \eta^{\mu\nu} + i\,\hbar(\beta + 2\varepsilon)\gamma^2 p^\mu p^\nu$$ [3]

- Defining (unphysical) canonical variables

$$p_0^\mu = -i\hbar \frac{\partial}{\partial x_{0\,\mu}}, \quad [x_0^\mu, p_0^\nu] = i\hbar\eta^{\mu\nu}$$

- Most general expression of the position and momentum operators

$$x^\mu = x_0^\mu - \alpha\gamma^2 p_0^\rho p_{0\rho} x_0^\mu + \underbrace{\beta\gamma^2 p_0^\mu p_0^\rho x_{0\rho} + \xi\hbar\gamma^2 p_0^\mu}_{\textit{breaks isotropy}},$$

$$p^\mu = p_0^\mu \left(1 + \varepsilon\gamma^2 p_0^\rho p_{0\rho}\right)$$

- Non-commutative spacetime

$$[x^\mu, x^\nu] = i\hbar\gamma^2 \frac{-2\alpha + \beta}{1 + (\varepsilon - \alpha)\gamma^2 p^\rho p_\rho} \left(x^\mu p^\nu - x^\nu p^\mu\right)$$

3. C. Quesne and V. M. Tkachuk, Czech. J. Phys. **56**, 1269 (2006) [quant-ph/0612093].

Poincaré group

- Lorentz generators

$$M^{\mu\nu} = p^\mu x^\nu - p^\nu x^\mu = \left[1 + (\varepsilon - \alpha)\gamma^2 p_0^\rho p_{0\,\rho}\right]\left(x_0^\mu p_0^\nu - x_0^\nu p_0^\mu\right)$$

- Poincaré algebra

$$[x^\mu, M^{\nu\rho}] = i\hbar[1 + (\varepsilon - \alpha)\gamma^2 p^\rho p_\rho]\left(x^\nu \delta^{\mu\rho} - x^\rho \delta^{\mu\nu}\right) + i\hbar 2(\varepsilon - \alpha)\gamma^2 p^\mu M^{\nu\rho}$$

$$[p^\mu, M^{\nu\rho}] = i\hbar[1 + (\varepsilon - \alpha)\gamma^2 p^\rho p_\rho]\left(p^\nu \delta^{\mu\rho} - p^\rho \delta^{\mu\nu}\right)$$

$$[M^{\mu\nu}, M^{\rho\sigma}] = i\hbar\left(1 + (\varepsilon - \alpha)\gamma^2 p^\rho p_\rho\right)\left(\eta^{\mu\rho} M^{\nu\sigma} - \eta^{\mu\sigma} M^{\nu\rho} - \eta^{\nu\rho} M^{\mu\sigma} + \eta^{\nu\sigma} M^{\mu\rho}\right)$$

- Special line in parameter space, which preserves the Poincaré algebra

$$\alpha = \varepsilon$$

- Position and momentum operators and their commutation relations

$$[x^\mu, p^\nu] = i\hbar\left(\eta^{\mu\nu} + 2\alpha\gamma^2 p^\mu p^\nu\right) \qquad \text{RGUP!}$$

$$[x^\mu, x^\nu] = -2i\hbar\alpha\gamma^2\left(x^\mu p^\nu - x^\nu p^\mu\right) \qquad \text{NC spacetime}$$

Casimir operators and Dispersion relation

- Casimir operator and Dispersion relation
  $E^2 = (pc)^2 + (mc^2)^2 \;\Rightarrow\; p^\rho p_\rho = -(mc)^2 \Longrightarrow p_0^\rho p_{0\rho}(1 + 2\alpha\gamma^2 p_0^\sigma p_{0\sigma}) = -(mc)^2$
- Fourth order differential equation using $p_0^2 = -\hbar^2 \partial^\rho \partial_\rho = \hbar^2 \Box_0$

$$\Box_0(1 + 2\alpha\gamma^2\hbar^2\Box_0) = -\frac{(mc)^2}{\hbar^2} \quad \text{where} \quad \Box_0 = \frac{\partial}{\partial x_0^\mu}\frac{\partial}{\partial x_{0\mu}}$$

- Not solvable using spherical harmonics
- So we solve for $p_0^\rho p_{0\rho}$

$$\begin{aligned} p_0^\rho p_{0\rho} &= -\frac{1}{4\alpha\gamma^2} \pm \sqrt{\frac{1}{(4\alpha\gamma^2)^2} - \frac{(mc)^2}{2\alpha\gamma^2}} \\ &\simeq -(m\,c)^2 - 2\alpha\gamma^2(m\,c)^4 - \mathcal{O}\left(\gamma^4\right) \end{aligned}$$

Klein-Gordon and Dirac equations

- Energy levels

$$E_{0\,N} = \left(m\,c^2 + \alpha\gamma^2 m^3 c^4\right) - \frac{\kappa^2\left(m\,c^2\right)}{2N^2} + \frac{3\kappa^4\left(mc^2\right)}{8N^4} + \frac{3\kappa^4\left(\alpha\gamma^2 m^3 c^4\right)}{8N^4} - \frac{\kappa^2\left(\alpha\gamma^2 m^3 c^4\right)}{2N^2}$$

- We find that all the RGUP corrections to the physical energy cancel

$$E_N = \left(m\,c^2\right) - \frac{\kappa^2\left(m\,c^2\right)}{2N^2} + \frac{\kappa^4\left(mc^2\right)}{8N^4}$$

Schrödinger equation

- Taking the non-relativistic limit of the Klein-Gordon equation but keeping the relativistic kinetic energy corrections, and the RGUP corrections

$$E_0 = mc^2 \left(1 + \underbrace{\alpha\gamma^2 (mc)^2}_{GUP}\right) + \frac{\vec{p}_0^2}{2m}\left(1 - \underbrace{\frac{1}{2}\alpha\gamma^2 (mc)^2}_{GUP}\right) - \underbrace{\frac{\vec{p}_0^4}{8m^3c^2}}_{Relativistic}\left(1 - \underbrace{3\alpha\gamma^2 (mc)^2}_{GUP+Relativistic}\right)$$

- Differential form

$$i\hbar\frac{\partial}{\partial t_0}\Psi(t_0,\vec{x}_0) = \Big[mc^2\left(1+\alpha\gamma^2 m^2 c^2\right) + \frac{(-i\hbar)^2}{2m}\left(1-\frac{1}{2}\alpha\gamma^2 m^2c^2\right)\nabla_0^2 - \frac{(-i\hbar)^4}{8m^3c^2}\left(1-3\alpha\gamma^2m^2c^2\right)\nabla_0^4 + V(\vec{x})\Big]\Psi(t_0,\vec{x}_0)$$

Applications

- Harmonic oscillator Energy levels

$$E_n = \hbar\omega\left(n+\frac{1}{2}\right)\left(1-\underbrace{\frac{1}{2}\alpha\gamma^2 m^2 c^2}_{GUP}\right) \underbrace{-\frac{\hbar^2\omega^2}{32mc^2}}_{Relativistic}\left(1-\underbrace{4\alpha\gamma^2 m^2 c^2}_{Relativistic+GUP}\right)[5n(n+1)+3]$$

- Order of the corrections for the energy of Landau levels [4] [5]

$$\frac{\Delta E_n}{E_n} = -\frac{3}{4}\alpha\gamma^2 m^2 c^2 \sim \alpha\, 10^{-44}$$

- Bounds on the GUP parameters

$$\alpha\, 10^{-44} \leq 10^{-3} \quad \Rightarrow \quad \alpha \leq 10^{41}$$

4. J. W. G. Wildöer, C. J. P. M. Harmans, and H. van Kempen, Phys. Rev. **B55**, R16013
5. L.-J. Yin, Y. Zhang, J.-B. Qiao, S.-Y. Li, and L. He, Phys. Rev. **B93**, 125422 (2016)

Summary and outlook

- Summary
  1. We got a manifestly Lorentz covariant GUP with the respective minimum length
  2. Theories with Relativistic Generalized Uncertainty Principle give rise to non-commutative spacetime at high energies.
  3. For certain relationship between the parameters of the theory the Poincaré group remains unmodified
  4. Applied to existing Quantum experiments give prediction for the parameter 10 orders of magnitude lower than previous works
- Outlook
  1. Formulating Quantum Field Theory with minimum length.
  2. Exploring potential experimental signatures. And getting stronger bounds on the scale of Quantum Gravity effects.
  3. Extending the Relativistic Generalized Uncertainty Principle to curved spacetimes, with applications to Quantum Gravity and Cosmology.

Thank you for your attention !